Symmetric third-order recurring sequences, Chebyshev polynomials, and Riordan arrays

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Abstract

We study a family of symmetric third-order recurring sequences with the aid of Riordan arrays and Chebyshev polynomials. Formulas involving both Chebyshev polynomials and Fibonacci numbers are established. The family of sequences defined by the product of consecutive terms of the first family of sequences is also studied, and links to the Chebyshev polynomials are again established, including continued fraction expressions. A multiplicative result is established relating Chebyshev polynomials to sequences of doubled Chebyshev polynomials. Links to a special Catalan related Riordan array are explored.

1 Introduction

In a posting to `seqfan`, the forum for contributors to the OEIS [7], Richard Guy invited comments on the 'symmetric third-order recurring sequences'. This note explores some links between these sequences and Riordan arrays [6, 9]. Riordan arrays provide a convenient vehicle for exploring many integer sequences, and can provide insight into the algebraic relationships that exist between them. These can then often be translated into binomial identities.

We can illustrate the context of this note by taking a simple example, based on the Fibonacci numbers F_n, where $F_0 = 0$, $F_1 = 1$. The sequence of "golden rectangle numbers" is the sequence of products F_nF_{n+1}. Thus we have (where we start at index 1)

n	1	2	3	4	5	6	7	...
F_n	1	1	2	3	5	8	13	...
F_{n+1}	1	2	3	5	8	13	21	...
F_nF_{n+1}	1	2	6	15	40	104	273	...

The generating function of the product sequence is given by

$$\frac{x}{1-2x-2x^2+x^3} = \frac{x}{(1+x)(1-3x+x^2)}.$$

Thus the product sequence satisfies a "symmetric third-order recurrence". It is also equal to an alternating sum of the terms of the sequence with generating function

$$\frac{x}{1-3x+x^2}.$$

Thus

$$F_n F_{n+1} = \sum_{k=0}^{n} (-1)^{n-k} F_{2k},$$

since F_{2n} is the sequence with g.f. $\frac{x}{1-3x+x^2}$. This motivates us to rearrange the table above in a "zig-zag" fashion.

n	1	2	3	4	5	6	7	...
v_{n+1}	1	2	2	5	5	13	13	...
u_n	1	1	3	3	8	8	21	...
$F_n F_{n+1}$	1	2	6	15	40	104	273	...

Thus the sequence $F_n F_{n+1}$ has another factorization, involving doubled sequences. The sequence u_n or $1, 1, 3, 3, 8, 8, \ldots$ has g.f.

$$\frac{1+x}{1-3x^2+x^4},$$

while the sequence v_n or $1, 1, 2, 2, 5, 5, \ldots$ has g.f.

$$\frac{(1+x)(1-x^2)}{1-3x^2+x^4}.$$

We note four points.

1. The "base" sequence F_n has generating function

$$\frac{x}{1-x-x^2} = \frac{x(1+x-x^2)}{1-3x^2+x^4}.$$

2. The elements F_{2n+2} of the sequence $1, 3, 8, 13, \ldots$ are values of Chebyshev polynomials of the second kind:

$$F_{2n+2} = U_n(3/2).$$

   Thus

$$F_{n+1} F_{n+2} = \sum_{k=0}^{n} (-1)^{n-k} U_k(3/2).$$

3. The sequence $1, 2, 5, 13, \ldots$ represents the first differences of $1, 3, 8, 21, \ldots$. Thus the product sequence has another factorization $u_n v_{n+1}$ involving the doubling of a (Chebyshev) sequence and its first differences.

4. We have

$$\sum_{i=0}^{n} F_i F_{i+1} = \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{i-j} F_{2j} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} F_{2(n-2k)} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} U_{n-2k-1}(3/2).$$

The goal of this paper is to show that *all* sequences with generating functions of the form

$$\frac{1}{1 - kx - kx^2 + x^4}$$

enjoy similar properties to those of the products of consecutive Fibonacci numbers: they are all the product of consecutive terms of a related "base" sequence, and both sequences (base and product sequence) may be expressed in terms of Chebyshev polynomials. We employ the techniques of Riordan arrays to establish other interesting properties of these sequences.

The plan of the paper is as follows:

1. This Introduction
2. Notation and recall of results
3. The polynomial sequences $a(n;r)$, product sequences $b(n;r)$ and Riordan arrays
4. $a(n;r)$ and Chebyshev polynomials
5. $b(n;r)$ and Chebyshev polynomials
6. From $b(n;r)$ to $a(n;r)$
7. A Chebyshev product result
8. Additional Riordan array factorizations
9. A left inverse
10. The matrix $\left(\frac{1-x}{1+x^4}, \frac{x}{1+x^2}\right)$
11. Appendix
12. Acknowledgements

2 Notation and recall of results

For an integer sequence a_n, that is, an element of $\mathbf{Z}^{\mathbf{N}}$, the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is called the *ordinary generating function* or g.f. of the sequence. a_n is thus the coefficient of x^n in this series. We denote this by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ is the n-th Catalan number.

The *Riordan group* [6], [9], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1 x + g_2 x^2 + \ldots$ and $f(x) = f_1 x + f_2 x^2 + \ldots$ where $f_1 \neq 0$ [9]. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) * (h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

A Riordan array of the form $(g(x), x)$, where $g(x)$ is the generating function of the sequence a_n, is called the *sequence array* of the sequence a_n. Its general term is a_{n-k}. Such arrays are also called *Appell* arrays as they form the elements of the Appell subgroup.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)'$ is an integer sequence with ordinary generating function $\mathcal{A}\ (x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbf{Z}^{\mathbf{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbf{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \longrightarrow (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

Example 1. The so-called *binomial matrix* $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$, and hence as an array coincides with Pascal's triangle. More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

Example 2. If a_n has generating function $g(x)$, then the generating function of the sequence

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-2k}$$

is equal to

$$\frac{g(x)}{1-x^2} = \left(\frac{1}{1-x^2}, x\right) \cdot g(x).$$

The row sums of the matrix (g, f) have generating function

$$(g, f) \cdot \frac{1}{1-x} = \frac{g(x)}{1-f(x)}$$

while the diagonal sums of (g, f) (sums of left-to-right diagonals in the North East direction) have generating function $g(x)/(1 - xf(x))$. These coincide with the row sums of the "generalized" Riordan array (g, xf). Thus the Fibonacci numbers F_{n+1} are the diagonal sums of the binomial matrix $\mathbf{B}$ given by $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$:

$$\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 2 & 1 & 0 & 0 & 0 & \ldots \\
1 & 3 & 3 & 1 & 0 & 0 & \ldots \\
1 & 4 & 6 & 4 & 1 & 0 & \ldots \\
1 & 5 & 10 & 10 & 5 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

while they are the row sums of the "generalized" or "stretched" (using the nomenclature of [1]) Riordan array $\left(\frac{1}{1-x}, \frac{x^2}{1-x}\right)$:

$$\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 2 & 0 & 0 & 0 & 0 & \ldots \\
1 & 3 & 1 & 0 & 0 & 0 & \ldots \\
1 & 4 & 3 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

We note that

$$\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 2 & 1 & 0 & 0 & 0 & \ldots \\
1 & 3 & 3 & 1 & 0 & 0 & \ldots \\
1 & 4 & 6 & 4 & 1 & 0 & \ldots \\
1 & 5 & 10 & 10 & 5 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)\left(\begin{array}{c}
1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ \vdots
\end{array}\right)=\left(\begin{array}{c}
1 \\ x+1 \\ (x+1)^2 \\ (x+1)^3 \\ (x+1)^4 \\ (x+1)^5 \\ \vdots
\end{array}\right)$$

while a second application of this matrix will send x^n to $(x+2)^n$.

To each proper Riordan matrix $(f_1 \neq 0)$

$$(g(x), f(x)) = (g(x), xh(x)) = (M_{n,k})_{n,k \geq 0}$$

there exist [2] two sequences $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)(\alpha_0 \neq 0)$ and $\xi = (\xi_0, \xi_1, \xi_2, \ldots)$ such that

1. Every element in column 0 can be expressed as a linear combination of all the elements in the preceding row, the coefficients being the elements of the sequence ξ, i.e.

$$M_{n+1,0} = \xi_0 M_{n,0} + \xi_1 M_{n,1} + \xi_2 M_{n,2} + \ldots$$

2. Every element $M_{n+1,k+1}$ not lying in column 0 or row 0 can be expressed as a linear combination of the elements of the preceding row, starting from the preceding column on, the coefficients being the elements of the sequence α, i.e.

$$M_{n+1,k+1} = \alpha_0 M_{n,k} + \alpha_1 M_{n,k+1} + \alpha_2 M_{n,2} + \ldots$$

The sequences α and ξ are called the *α-sequence* and the *ξ-sequence* of the Riordan matrix. It is customary to use the same symbols α and ξ as the names of the corresponding generating functions. The functions $g(x)$, $f(x)$, $\alpha(x)$ and $\xi(x)$ are connected as follows:

$$f(x) = x\alpha(f(x)), \qquad g(x) = \frac{M_{0,0}}{1 - x\xi(f(x))}.$$

The α-sequence is sometimes called the *A-sequence* of the array and then we write $A(x) = \alpha(x)$.

Many interesting examples of Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences, [7], [8]. Sequences are frequently referred to by their OEIS number. For instance, the matrix $\mathbf{B}$ is A007318.

It is often the case that we work with "generalized" Riordan arrays, where we relax some of the conditions above. Thus for instance [1] discusses the notion of the "stretched" Riordan array. In this note, we shall encounter "vertically stretched" arrays of the form (g, h) where now $h_0 = h_1 = 0$ with $h_2 \neq 0$. Such arrays are not invertible, but we may explore their left inversion. In this context, standard Riordan arrays as described above are called "proper" Riordan arrays. We note for instance that for any proper Riordan array (g, f), its diagonal sums are just the row sums of the vertically stretched array (g, xf) and hence have g.f. $g/(1 - xf)$.

Each Riordan array $(g(x), f(x))$ has bi-variate generating function given by

$$\frac{g(x)}{1 - yf(x)}.$$

For instance, the binomial matrix $\mathbf{B}$ has generating function

$$\frac{\frac{1}{1-x}}{1 - y\frac{x}{1-x}} = \frac{1}{1 - x(1+y)}.$$

For a sequence $a_0, a_1, a_2, \ldots$ with g.f. $g(x)$, the "aeration" of the sequence is the sequence $a_0, 0, a_1, 0, a_2, \ldots$ with interpolated zeros. Its g.f. is $g(x^2)$. The sequence $a_0, a_0, a_1, a_1, a_2, \ldots$ is called the "doubled" sequence. It has g.f. $(1 + x)g(x^2)$.

The Chebyshev polynomials of the second kind, $U_n(x)$, are defined by

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-1)^k (2x)^{n-2k}$$

Alternatively,

$$U_n(x) = \sum_{k=0}^{n} \binom{\frac{n+k}{2}}{k} (-1)^{\frac{n-k}{2}} \frac{1 + (-1)^{n-k}}{2} (2x)^k.$$

They satisfy the three-term recurrence

$$U_{n+1} = 2xU_n(x) - U_{n-1}(x), \quad U_0(x) = 1, \quad U_1(x) = 2x. \tag{1}$$

The generating function for $U_n(x)$ is given by

$$\sum_{k=0}^{\infty} U_n(x) y^n = \frac{1}{1 - 2xy + y^2}. \tag{2}$$

We note that the coefficient array of the polynomials $U_n(x/2)$ which begin $1, x, x^2 - 1, x^3 - 2x, x^4 - 3x^2 + 1, \ldots,$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & -2 & 0 & 1 & 0 & 0 & \ldots \\ 1 & 0 & -3 & 0 & 1 & 0 & \ldots \\ 0 & 3 & 0 & -4 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is the Riordan array

$$\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right).$$

This follows since

$$\frac{\frac{1}{1+x^2}}{1 - y\frac{x}{1+x^2}} = \frac{1}{1 - xy + y^2} = \frac{1}{1 - 2\frac{x}{2}y + y^2}.$$

This matrix is A049310. Thus

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & -2 & 0 & 1 & 0 & 0 & \ldots \\ 1 & 0 & -3 & 0 & 1 & 0 & \ldots \\ 0 & 3 & 0 & -4 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ x^2 - 1 \\ x^3 - 2x \\ x^4 - 3x^2 + 1 \\ x^5 - 4x^3 + 3x \\ \vdots \end{pmatrix}.$$

Similarly, the polynomials $U_n\left(\frac{x+2}{2}\right)$ have coefficient array given by

$$\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right) \cdot \mathbf{B}^2 = \left(\frac{1}{(1-x)^2}, \frac{x}{(1-x)^2}\right).$$

In matrix terms, this gives us

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 3 & 4 & 1 & 0 & 0 & 0 & \dots \\ 4 & 10 & 6 & 1 & 0 & 0 & \dots \\ 5 & 20 & 21 & 8 & 1 & 0 & \dots \\ 6 & 35 & 56 & 36 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)\left(\begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ \vdots \end{array}\right)=\left(\begin{array}{c} 1 \\ x+2 \\ x^2+4x+3 \\ x^3+6x^2+10x+4 \\ x^4+8x^3+21x^2+20x+5 \\ x^5+10x^4+36x^3+56x^2+35x+6 \\ \vdots \end{array}\right).$$

This coefficient matrix is A078812, whose general term is $\binom{n+k+1}{2k+1}$. The polynomials so defined are the Morgan-Voyce polynomials [10]

$$M(n,x)=\sum_{k=0}^{n}\binom{n+k+1}{2k+1}x^k.$$

The Morgan-Voyce polynomials $m(n,x)$ are defined by

$$m(n,x)=\sum_{k=0}^{n}\binom{n+k}{2k}x^k=\sum_{k=0}^{n}\binom{n+k}{n-k}x^k.$$

The coefficient array for these polynomials, which has general term $\binom{n+k}{2k}$, is the Riordan array

$$\left(\frac{1}{1-x},\frac{x}{(1-x)^2}\right).$$

This is A085478. For this, we have

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 5 & 1 & 0 & 0 & \dots \\ 1 & 10 & 15 & 7 & 1 & 0 & \dots \\ 1 & 15 & 35 & 28 & 9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)\left(\begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ \vdots \end{array}\right)=\left(\begin{array}{c} 1 \\ x+1 \\ x^2+3x+1 \\ x^3+5x^2+6x+1 \\ x^4+7x^3+15x^2+10x+1 \\ x^5+9x^4+28x^3+35x^2+15x+1 \\ \vdots \end{array}\right).$$

The Morgan-Voyce polynomials can be defined by a three-term recurrence as follows:

$$m(0,x)=1,\quad m(1,x)=1+x,\quad m(n,x)=(x+2)m(n-1,x)-m(n-2,x) \tag{3}$$

and

$$M(0,x)=1,\quad M(1,x)=2+x,\quad M(n,x)=(x+2)M(n-1,x)-M(n-2,x). \tag{4}$$

In the sequel, we shall use the notation $i=\sqrt{-1}$.

3 The polynomial sequences $a(n;r)$, product sequences $b(n;r)$ and Riordan arrays

We wish to study sequences of polynomials of the form

$$a(n;r)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n-k}{k}r^{\lfloor \frac{n-2k}{2}\rfloor},$$

with generating function

$$g_a(x;r)=\frac{1+x-x^2}{1-(r+2)x^2+x^4}$$

and the related sequences $b(n;r)$ with generating function

$$g_b(x;r)=\frac{1}{1-(r+1)x-(r+1)x^2+x^3}.$$

Examples of these sequences in the OEIS [7] are given below.

r	$a(n;r)$	$b(n;r)$
0	A157329	A008619
1	A000045$(n+1)$	A001654$(n+1)$
2	A002530$(n+1)$	A109437$(n+1)$
3	A136211$(n+1)$	A099025
4	A041011	A084158$(n+1)$
5	A152119$(n+1)$	A157335

In each of the cases above, we have

$$b(n;r)=a(n;r)a(n+1;r).$$

Elements of the family $\{a(n;r)\}$ are images of the doubled sequences $1,1,r,r,r^2,r^2,\ldots$ with generating function $\frac{1+x}{1-rx^2}$ by the Riordan array

$$\left(\frac{1}{1-x^2},\frac{x}{1-x^2}\right)$$

(closely related to the Chebyshev polynomials). This is so since we have

$$\begin{aligned}\left(\frac{1}{1-x^2},\frac{x}{1-x^2}\right)\cdot\frac{1+x}{1-rx^2} &= \frac{1}{1-x^2}\frac{1+\frac{x}{1-x^2}}{1-r\left(\frac{x}{1-x^2}\right)^2}\\ &= \frac{1+x-x^2}{1-(r+2)x^2+x^4},\end{aligned}$$

the generating function of $a(n;r)$. An immediate consequence of this is the fact that

$$\begin{aligned} a(n;r) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} r^{\lfloor \frac{n-2k}{2} \rfloor} \\ &= \sum_{k=0}^{n} \binom{k}{n-k} r^{\lfloor \frac{2k-n}{2} \rfloor} \\ &= \sum_{k=0}^{n} \binom{\frac{n+k}{2}}{k} \frac{1+(-1)^{n-k}}{2} r^{\lfloor \frac{k}{2} \rfloor}. \end{aligned}$$

$a(n;r)$ thus represents the diagonal sums of the number triangle with general term $\binom{n}{k} r^{\lfloor \frac{n-k}{2} \rfloor}$. It is clear that $a(n;r)$ is a polynomial in r. We can express $a(n;r)$ directly in terms of ascending powers of r by noticing that the coefficient array of the polynomials $a(n;r)$, which begins

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 3 & 4 & 1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

is the (generalized) Riordan array

$$\left(\frac{1+x-x^2}{(1-x^2)^2}, \frac{x^2}{(1-x^2)^2} \right)$$

with general term

$$\binom{\lfloor \frac{n+2k+1}{2} \rfloor}{\lfloor \frac{n-2k}{2} \rfloor}.$$

(A proof of this assertion is indicated in the Appendix). Thus we have

$$a(n;r) = \sum_{k=0}^{n} \binom{\lfloor \frac{n+2k+1}{2} \rfloor}{\lfloor \frac{n-2k}{2} \rfloor} r^k. \tag{5}$$

This translates the algebraic identity

$$\left(\frac{1+x-x^2}{(1-x^2)^2}, \frac{x^2}{(1-x^2)^2} \right) \cdot \frac{1}{1-rx} = \frac{1+x-x^2}{1-(r+2)x^2+x^4}.$$

The family of polynomials $a(n;r)$ begins

$$1, 1, r+1, r+2, r^2+3r+1, r^2+4r+3, r^3+5r^2+6r+1, \dots$$

We note that if we define a family of polynomials $\mathcal{P}(n,x)$ as

$$\mathcal{P}(0,x) = 1, \quad \mathcal{P}(1,x) = 1, \quad \mathcal{P}(n,x) = \mathcal{P}(n-1,x) + x\mathcal{P}(n-2,x), \quad n > 1$$

then we have

$$a(n,r) = r^{\lfloor \frac{n}{2} \rfloor} \mathcal{P}\left(n, \frac{1}{r}\right).$$

We further have

$$\begin{aligned}
\frac{1+x-x^2}{1-(r+2)x^2+x^4} &= \left(\frac{1}{1-x^2}, \frac{x}{1-x^2}\right) \cdot \frac{1+x}{1-rx^2} \\
&= \left(\frac{1}{1-x^2}, \frac{x}{1-x^2}\right) \cdot (1+x, x^2) \cdot \frac{1}{1-rx} \\
&= \left(\frac{1+x-x^2}{(1-x^2)^2}, \frac{x^2}{(1-x^2)^2}\right) \cdot \frac{1}{1-rx}
\end{aligned}$$

where the last matrix is not invertible (due to the factor $(1+x, x^2)$).

We can describe $a(n;r)$ in terms of the Morgan-Voyce polynomials as follows:

Proposition 3. *We have*

$$a(2n;r) = m(n,r), \qquad a(2n+1;r) = M(n,r).$$

Proof. We have

$$\begin{aligned}
a(2n;r) &= \sum_{k=0}^{n} \binom{2n-k}{k} r^{n-k} \\
&= \sum_{k=0}^{n} \binom{2n-(n-k)}{n-k} r^{k} \\
&= \sum_{k=0}^{n} \binom{n+k}{n-k} r^{k}.
\end{aligned}$$

The case of $a(2n+1;r)$ follows in like manner. □

Corollary 4.

$$a(n;r) = m(n/2, r)\frac{1+(-1)^n}{2} + M((n-1)/2, r)\frac{1-(-1)^n}{2}.$$

The main result of this paper is that the observation

$$b(n;r) = a(n;r)a(n+1;r)$$

holds in general.

Theorem 5. *Let $a(n;r)$ be the sequence of polynomials with generating function*

$$g_a(x;r) = \frac{1+x-x^2}{1-(r+2)x^2+x^4}.$$

Also let $b(n;r)$ be the sequence of polynomials with generating function

$$g_b(x;r) = \frac{1}{1-(r+1)x-(r+1)x^2+x^3}.$$

Then we have, for all $n, r \in \mathbb{Z}$,

$$b(n;r) = a(n;r)a(n+1;r).$$

Proof. We have

$$g_a(x;r) = \frac{1+x-x^2}{1-(r+2)x^2+x^4} = A(x)$$

so that

$$A(x)-(r+2)x^2A(x)+x^4A(x) = 1+x-x^2.$$

Extracting the coefficient of x^{n+4}, we obtain

$$[x^{n+4}]A(x)-(r+2)[x^{n+2}]A(x)+[x^n]A(x) = [x^{n+4}](1+x-x^2)$$

$$A_{n+4}-(r+2)A_{n+2}+A_n = 0. \tag{6}$$

By direct inspection, the first terms of the sequence $A_n = a(n;r)$ are

$$1, 1, 1+r, 2+r, 1+3r+r^2, 3+4r+r^2, \ldots.$$

We now set $B_n = A_nA_{n+1}$ and $T_n = A_nA_{n+3}$, and try to determine the corresponding generating functions. From recurrence (6), we have

$$\begin{aligned} A_{n+4} &= (r+2)A_{n+2}-A_n \\ A_{n+5} &= (r+2)A_{n+3}-A_{n+1}. \end{aligned}$$

Multiplying, we obtain

$$A_{n+4}A_{n+5} = (r+2)^2A_{n+2}A_{n+3}-(r+2)A_{n+1}A_{n+2}-(r+2)A_nA_{n+3}+A_nA_{n+1}.$$

This can be written as:

$$B_{n+4} = (r+2)^2B_{n+2}-(r+2)B_{n+1}-(r+2)T_n+B_n. \tag{7}$$

Now multiplying equation (6) by A_{n+3}, we obtain

$$A_{n+3}A_{n+4} = (r+2)A_{n+2}A_{n+3}-A_nA_{n+3},$$

or

$$B_{n+3} = (r+2)B_{n+2}-T_n. \tag{8}$$

We can now use the method of coefficients [5] to pass from the recurrences to the corresponding generating functions $B(x)$ and $T(x)$. In order to do so, we note that the coefficients of the sequence generated by $B(x)$ begin:

$$1, 1+r, 2+3r+r^2, 2+7r+5r^2+r^3, \ldots.$$

Equation (7) becomes

$$\begin{aligned}&\frac{B(x)-1-(1+r)x-(2+3r+r^2)x^2-(2+7r+5r^2+r^3)x^3}{x^4} \\ &= (r+2)\frac{B(x)-1-(1+r)x}{x^2}-(r+2)\frac{B(x)-1}{x}-(r+2)T(x)+B(x)\end{aligned} \tag{9}$$

while Equation (8) becomes:

$$\frac{B(x)-1-(1+r)x-(2+3r+r^2)x^2}{x^3}=(r+2)\frac{B(x)-1-(1+r)x}{x^2}-T(x). \tag{10}$$

We now have two linear equations for the two unknowns $B(x)$ and $T(x)$. Solving, we find that

$$B(x)=\frac{1}{1-(r+1)x-(r+1)x^2+x^3}=g_b(x;r)$$

as required. □

Corollary 6.

$$T(x)=\frac{(r+2)-x}{1-(r+1)x-(r+1)x^2+x^3}$$

is the generating function of the product $a(n;r)a(n+3;r)$.

The author is indebted to an anonymous reviewer for the form of this proof.

We note that this theorem establishes the binomial identity

$$\sum_{j=0}^{k}\binom{\lfloor\frac{n+1+2j}{2}\rfloor}{\lfloor\frac{n-2j}{2}\rfloor}\binom{\lfloor\frac{n+2+2(k-j)}{2}\rfloor}{\lfloor\frac{n+1-2(k-j)}{2}\rfloor}=\sum_{j=0}^{n}(-1)^{n-j}\binom{j+k+1}{2k+1} \tag{11}$$

where the left hand side follows from the Cauchy form of the product of two polynomials and the right hand side is an expression for the general term of the $b(n;r)$ coefficient array $\left(\frac{1}{(1-x)(1-x^2)},\frac{x}{(1-x)^2}\right)$.

4 $a(n;r)$ and Chebyshev polynomials

We can express $a(n;r)$ in terms of Chebyshev polynomials. This is the content of the following proposition.

Proposition 7.

$$a(n;r)=i^n\left\{\frac{\sqrt{r}+1}{2\sqrt{r}}U_n\left(-i\frac{\sqrt{r}}{2}\right)+\frac{\sqrt{r}-1}{2\sqrt{r}}U_n\left(i\frac{\sqrt{r}}{2}\right)\right\}. \tag{12}$$

Proof. We recall that the g.f. of $a(n;r)$ is $\frac{1+x-x^2}{1-(r+2)x^2+x^4}$. Now

$$\begin{aligned}
\frac{1+x-x^2}{1-(r+2)x^2+x^4} &= \frac{1-x^2}{1-(r+2)x^2+x^4}+\frac{x}{1-(r+2)x^2+x^4}\\
&= \frac{1}{2}\left\{\frac{1}{1-\sqrt{r}x-x^2}+\frac{1}{1+\sqrt{r}x-x^2}\right\}+\frac{1}{2\sqrt{r}}\left\{\frac{1}{1-\sqrt{r}x-x^2}-\frac{1}{1+\sqrt{r}x-x^2}\right\}\\
&= \left(\frac{1}{2}+\frac{1}{2\sqrt{r}}\right)\frac{1}{1-\sqrt{r}x-x^2}+\left(\frac{1}{2}-\frac{1}{2\sqrt{r}}\right)\frac{1}{1+\sqrt{r}x-x^2}\\
&= \frac{\sqrt{r}+1}{2\sqrt{r}}\frac{1}{1-\sqrt{r}x-x^2}+\frac{\sqrt{r}-1}{2\sqrt{r}}\frac{1}{1+\sqrt{r}x-x^2}.
\end{aligned}$$

The assertion follows from this, given Eq. (2). □

Corollary 8.

$$a(n;r)=i^nU_n\left(i\frac{\sqrt{r}}{2}\right)\left\{(-1)^n\frac{\sqrt{r}+1}{2\sqrt{r}}+\frac{\sqrt{r}-1}{2\sqrt{r}}\right\}. \tag{13}$$

Proof. We have $U_n(-x)=(-1)^nU_n(x)$. The result follows from this. □

This now allows us to characterize the product $a(n;r)a(n+1;r)$ in terms of Chebyshev polynomials.

Proposition 9.

$$a(n;r)a(n+1;r)=i(-1)^{n+1}\frac{U_n\left(i\frac{\sqrt{r}}{2}\right)U_{n+1}\left(i\frac{\sqrt{r}}{2}\right)}{\sqrt{r}}. \tag{14}$$

Proof.

$$\begin{aligned}
a(n;r)a(n+1;r) &= i(-1)^nU_n\left(i\frac{\sqrt{r}}{2}\right)U_{n+1}\left(i\frac{\sqrt{r}}{2}\right)\left\{(-1)^n\frac{\sqrt{r}+1}{2\sqrt{r}}+\frac{\sqrt{r}-1}{2\sqrt{r}}\right\}\cdot\\
&\qquad\left\{-(-1)^n\frac{\sqrt{r}+1}{2\sqrt{r}}+\frac{\sqrt{r}-1}{2\sqrt{r}}\right\}\\
&= i(-1)^nU_n\left(i\frac{\sqrt{r}}{2}\right)U_{n+1}\left(i\frac{\sqrt{r}}{2}\right)\frac{1}{4r}(-(\sqrt{r}+1)^2+(\sqrt{r}-1)^2)\\
&= i(-1)^nU_n\left(i\frac{\sqrt{r}}{2}\right)U_{n+1}\left(i\frac{\sqrt{r}}{2}\right)\frac{1}{4r}.-4\sqrt{r}\\
&= i(-1)^{n+1}\frac{U_n\left(i\frac{\sqrt{r}}{2}\right)U_{n+1}\left(i\frac{\sqrt{r}}{2}\right)}{\sqrt{r}}.
\end{aligned}$$

□

The following proposition provides a link between the polynomials $a(n;r)$ and continued fractions.

Proposition 10.

$$a(n+2;r) = a(n+1;r)\frac{r(1+(-1)^n)+1-(-1)^n}{2} + a(n;r) \tag{15}$$

Proof. We have

$$\begin{aligned}
a(n+2;r) &= i^{n+2}U_{n+2}\left(\frac{i\sqrt{r}}{2}\right)\left\{(-1)^{n+2}\frac{\sqrt{r}+1}{2\sqrt{r}} + \frac{\sqrt{r}-1}{2\sqrt{r}}\right\} \\
&= -i^n\left(2i\frac{\sqrt{r}}{2}U_{n+1}\left(\frac{i\sqrt{r}}{2}\right) - U_n\left(\frac{i\sqrt{r}}{2}\right)\right)\left\{(-1)^{n+2}\frac{\sqrt{r}+1}{2\sqrt{r}} + \frac{\sqrt{r}-1}{2\sqrt{r}}\right\} \\
&= -i^{n+1}\sqrt{r}U_{n+1}\left(\frac{i\sqrt{r}}{2}\right)\left\{(-1)^n\frac{\sqrt{r}+1}{2\sqrt{r}} + \frac{\sqrt{r}-1}{2\sqrt{r}}\right\} + \\
&\qquad i^nU_n\left(\frac{i\sqrt{r}}{2}\right)\left\{(-1)^n\frac{\sqrt{r}+1}{2\sqrt{r}} + \frac{\sqrt{r}-1}{2\sqrt{r}}\right\} \\
&= a(n+2;r)\frac{-\sqrt{r}\left\{(-1)^n\frac{\sqrt{r}+1}{2\sqrt{r}} + \frac{\sqrt{r}-1}{2\sqrt{r}}\right\}}{\left\{(-1)^{n+1}\frac{\sqrt{r}+1}{2\sqrt{r}} + \frac{\sqrt{r}-1}{2\sqrt{r}}\right\}} + a(n;r) \\
&= a(n+1;r)\frac{r(1+(-1)^n)+1-(-1)^n}{2} + a(n;r).
\end{aligned}$$

□

Inspired by [4], we now define a family of rational functions $R(n,r)$ as follows:

$$R(n,r) = \begin{cases} 1 & \text{if } n=0 \\ 1 & \text{if } n=1 \\ \frac{1}{1+\frac{1}{r}R(n-1,r)} & \text{if } n>1 \end{cases}$$

The first elements of the family are

$$1, 1, \frac{r}{r+1}, \frac{r+1}{r+2}, \frac{r(r+2)}{r^2+3r+1}, \frac{r^2+3r+1}{r^2+4r+3}, \ldots$$

Proposition 11.

$$a(n;r) = \mathit{Denominator}(R(n,r))$$

Proof. Induction, using Eq. (15). □

Noting that

$$R(n,r) = \frac{i\sqrt{r}U_{n-1}\left(\frac{i\sqrt{r}}{2}\right)}{U_n\left(\frac{i\sqrt{r}}{2}\right)}, \quad n>0,$$

we have

$$a(n;r) = \text{Denominator}\left(\frac{i\sqrt{r}U_{n-1}\left(\frac{i\sqrt{r}}{2}\right)}{U_n\left(\frac{i\sqrt{r}}{2}\right)}\right). \tag{16}$$

Finally, we can use the expression

$$U_n(x) = 2^n \prod_{k=1}^{n} \left(x - \cos\left(\frac{k\pi}{n+1}\right)\right)$$

and Corollary **13** to give another characterization of $a(n;r)$.

Proposition 12.

$$a(n;r) = \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(r + 4\cos^2\left(\frac{k\pi}{n+1}\right)\right). \tag{17}$$

Proof. We begin with the case $n = 2m$. Then

$$\begin{aligned}
i^n U_n\left(\frac{i\sqrt{r}}{2}\right) &= i^n 2^n \prod_{k=1}^{n} \left(\frac{i\sqrt{r}}{2} - \cos\left(\frac{k\pi}{n+1}\right)\right) \\
&= i^n \prod_{k=1}^{n} \left(i\sqrt{r} - 2\cos\left(\frac{k\pi}{n+1}\right)\right) \\
&= i^n \prod_{k=1}^{m} \left(i\sqrt{r} - 2\cos\left(\frac{k\pi}{n+1}\right)\right)\left(i\sqrt{r} + 2\cos\left(\frac{k\pi}{n+1}\right)\right) \\
&= i^n \prod_{k=1}^{m} \left(-r - 4\cos^2\left(\frac{k\pi}{n+1}\right)\right) \\
&= i^{2m}(-1)^m \prod_{k=1}^{m} \left(r + 4\cos^2\left(\frac{k\pi}{n+1}\right)\right) \\
&= \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(r + 4\cos^2\left(\frac{k\pi}{n+1}\right)\right).
\end{aligned}$$

Note that in the product expansion we have used a first with last, second with next-to-last, etc coupling. We now turn to the case $n = 2m+1$. We obtain

$$\begin{aligned}
i^n U_n\left(\frac{i\sqrt{r}}{2}\right) &= i^n 2^n \prod_{k=1}^{n}\left(\frac{i\sqrt{r}}{2} - \cos\left(\frac{k\pi}{n+1}\right)\right) \\
&= i^n \prod_{k=1}^{n}\left(i\sqrt{r} - 2\cos\left(\frac{k\pi}{n+1}\right)\right) \\
&= i^n \left\{\prod_{k=1}^{m}\left(i\sqrt{r} - 2\cos\left(\frac{k\pi}{n+1}\right)\right)\left(i\sqrt{r} + 2\cos\left(\frac{k\pi}{n+1}\right)\right)\right\} \cdot \\
&\qquad \left(i\sqrt{r} - 2\cos\left(\frac{(m+\frac{1}{2})\pi}{2m+1}\right)\right) \\
&= i^n \prod_{k=1}^{m}\left(-r - 4\cos^2\left(\frac{k\pi}{n+1}\right)\right)\left(i\sqrt{r} - 2\cos\left(\frac{\pi}{2}\right)\right) \\
&= i^{2m+1}(-1)^m \prod_{k=1}^{m}\left(r + 4\cos^2\left(\frac{k\pi}{n+1}\right)\right)\left(i\sqrt{r} - 2\cos\left(\frac{\pi}{2}\right)\right) \\
&= \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor}\left(r + 4\cos^2\left(\frac{k\pi}{n+1}\right)\right)(-\sqrt{r}).
\end{aligned}$$

The result follows now from the fact that

$$a(n;r) = i^n U_n\left(i\frac{\sqrt{r}}{2}\right)\left\{(-1)^n \frac{\sqrt{r}+1}{2\sqrt{r}} + \frac{\sqrt{r}-1}{2\sqrt{r}}\right\}$$

where $(-1)^n \frac{\sqrt{r}+1}{2\sqrt{r}} + \frac{\sqrt{r}-1}{2\sqrt{r}}$ is the sequence

$$1, -\frac{1}{\sqrt{r}}, 1, -\frac{1}{\sqrt{r}}, 1, -\frac{1}{\sqrt{r}}, \ldots.$$

□

We note that the case $r = 5$ of this result is noted in A152119 (Roger L. Bagula and Gary W. Adamson).

5 $b(n;r)$ and Chebyshev polynomials

We can also express the second sequence family $b(n;r)$ in terms of Chebyshev polynomials.

Proposition 13.

$$b(n;r) = \sum_{k=0}^{n}(-1)^{n-k} U_k\left(\frac{r+2}{2}\right). \tag{18}$$

Proof. We have

$$\left(\frac{1}{(1-x)(1-x^2)},\frac{x}{(1-x)^2}\right)\cdot\frac{1}{1-rx} = \frac{1}{(1+x)(1-(r+2)x+x^2)}$$
$$= \frac{1}{1-(r+1)x-(r+1)x^2+x^3}$$

as an easy calculation will verify. This implies that for the sequence $b(n;r)$ with generating function

$$g_b(x;r)=\frac{1}{1-(r+1)x-(r+1)x^2+x^3}$$

we have

$$b(n;r)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^{n-k}U_k\left(\frac{r+2}{2}\right).$$

□

Corollary 14. *$b(n;r)$ is the first difference of the sequence with general term*

$$w(n;r)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}U_{n-2k}\left(\frac{r+2}{2}\right).$$

Proof. We have

$$\begin{aligned}
b(n;r) &= \sum_{k=0}^{n}(-1)^{n-k}U_k\left(\frac{r+2}{2}\right)\\
&= U_n\left(\frac{r+2}{2}\right)-U_{n-1}\left(\frac{r+2}{2}\right)+U_{n-2}\left(\frac{r+2}{2}\right)-\dots\\
&= U_n\left(\frac{r+2}{2}\right)+U_{n-2}\left(\frac{r+2}{2}\right)+\dots-U_{n-1}\left(\frac{r+2}{2}\right)-U_{n-3}\left(\frac{r+2}{2}\right)-\dots\\
&= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}U_{n-2k}\left(\frac{r+2}{2}\right)-\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}U_{n-2k-1}\left(\frac{r+2}{2}\right).
\end{aligned}$$

□

Corollary 15.

$$b(n;r)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\left\{U_{n-2k+1}\left(\frac{r+2}{2}\right)-(r+1)U_{n-2k}\left(\frac{r+2}{2}\right)\right\}.$$

Proof. We use the Chebyshev recurrence

$$U_{n+1}=2xU_n(x)-U_{n-1}(x),\quad U_0(x)=1,\quad U_1(x)=2x.$$

□

We note that the results above also follow directly from the observation that

$$g_b(x;r) = \frac{1}{(1+x)(1-(r+2)x+x^2)} = (1-x)\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-(r+2)x+x^2}.$$

Now the general term of the Riordan array

$$\left(\frac{1}{(1-x)(1-x^2)}, \frac{x}{(1-x)^2}\right) = \left(\frac{1+x}{(1-x^2)^2}, \frac{x}{(1-x)^2}\right) = \left(\frac{1}{1+x}, x\right)\left(\frac{1}{(1-x)^2}, \frac{x}{(1-x)^2}\right)$$

can be expressed as

$$\sum_{j=0}^{n-k}(-1)^j\binom{n+k-j+1}{n-k-j} = \sum_{j=0}^{n-k}(-1)^j\binom{n+k-j+1}{2k+1} = \sum_{j=0}^{n}(-1)^{n-j}\binom{j+k+1}{2k+1}.$$

This is A158909, which has row sums given by the product of consecutive Fibonacci numbers $F(n+1)F(n+2)$ and diagonal sums given by the Jacobsthal numbers A001045. The array begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 7 & 5 & 1 & 0 & 0 & \ldots \\ 3 & 13 & 16 & 7 & 1 & 0 & \ldots \\ 3 & 22 & 40 & 29 & 9 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We thus have

$$b(n;r) = \sum_{k=0}^{n}\sum_{j=0}^{n-k}(-1)^j\binom{n+k-j+1}{n-k-j}r^k = \sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^{n-j}\binom{j+k+1}{2k+1}r^k. \tag{19}$$

The polynomials $b(n;x)$ begin

$$1, x+1, x^2+3x+2, x^3+5x^2+7x+2, x^4+7x^3+16x^2+13x+3, \ldots$$

We note that

$$\begin{aligned} 1 &= 1 \\ x+1 &= 1\cdot(1+x) \\ x^2+3x+2 &= (1+x)(2+x) \\ x^3+5x^2+7x+2 &= (2+x)(1+3x+x^2) \\ x^4+7x^3+16x^2+13x+3 &= (1+3x+x^2)(3+4x+x^2)\ldots \end{aligned}$$

Alternative (and more esoteric) expressions can be found for $b(n;r)$ by taking other factorizations of the coefficient array. For instance,

$$\left(\frac{1}{(1-x)(1-x^2)}, \frac{x}{(1-x)^2}\right) = \left(\frac{1}{(1-x)^2}, \frac{x}{(1-x)^2}\right)\cdot\left(\frac{1+\sqrt{1+4x}}{2\sqrt{1+4x}}, x\right)$$

leads to the expression

$$b(n;r) = \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n+j+1}{n-j} \binom{2j-2k-1}{j-k} (-1)^{j-k} r^k.$$

6 From $b(n;r)$ to $a(n;r)$

We now show that we can find a transformation from the sequences $b(n;r)$ with generating function

$$g_b(x;r) = \frac{1}{1-(r+1)x-(r+1)x^2+x^3}$$

to the sequences $a(n;r)$ with generating function $g_a(x;r) = \frac{1+x-x^2}{1-(r+2)x^2+x^4}$. This follows since

$$\begin{aligned}
g_a(x;r) &= \frac{1+x-x^2}{1-(r+2)x^2+x^4} \\
&= \left(\frac{1+x-x^2}{(1-x^2)^2}, \frac{x^2}{(1-x^2)^2}\right) \cdot \frac{1}{1-rx} \\
&= \left(\frac{1+x-x^2}{(1-x^2)^2}, \frac{x^2}{(1-x^2)^2}\right) \cdot \left(\frac{1}{(1-x)(1-x^2)}, \frac{x}{(1-x)^2}\right)^{-1} \cdot \frac{1}{1-(r+1)x-(r+1)x^2+x^3} \\
&= \left(\frac{1+x-x^2}{(1-x^2)^2}, \frac{x^2}{(1-x^2)^2}\right) \cdot (c(-x)^2+xc(-x)^4, 1-c(-x)) \cdot \frac{1}{1-(r+1)x-(r+1)x^2+x^3} \\
&= (1+x+x^3-x^4, x^2) \cdot \frac{1}{1-(r+1)x-(r+1)x^2+x^3} \\
&= (1+x+x^3-x^4, x^2) \cdot g_b(x;r).
\end{aligned}$$

Here, we have used the notation

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}$$

for the generating function of the Catalan numbers A000108.

For instance, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 30 \\ 174 \\ 1015 \\ 5915 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \\ 6 \\ 29 \\ 35 \\ \vdots \end{pmatrix}$$

which displays the sequence with g.f. $\frac{1+x-x^2}{1-6x^2+x^4}$ and general term

$$a(n;4) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} 4^{\lfloor \frac{n-2j}{2} \rfloor}$$

as the image of the sequence with g.f. $\frac{1}{1-5x-5x^2+x^3}$ and general term

$$b(n;4) = a(n;4)a(n+1;4).$$

We thus have the following proposition.

Proposition 16. *The sequence $b(n;r)$ with generating function*

$$\frac{1}{1-(r+1)x-(r+1)x^2+x^3}$$

is transformed into the sequence $a(n;r)$ with generating function $\frac{1+x-x^2}{1-(r+2)x^2+x^4}$ and general term

$$a(n;r) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} r^{\lfloor \frac{n-2j}{2} \rfloor}$$

by the (generalized) Riordan array

$$(1+x+x^3-x^4, x^2).$$

We note that this transformation is not invertible. In terms of invertible transforms, we have the following result.

Proposition 17. *$a(n;r)$ is the image under the Riordan array $(1+x+x^3-x^4, x)$ of the aeration of $b(n;r)$.*

Proof. We have

$$\begin{aligned} \frac{1+x-x^2}{1-(r+2)x^2+x^4} &= \frac{1+x+x^3-x^4}{(1+x^2)(1-(r+2)x^2+x^4)} \\ &= (1+x+x^3-x^4, x) \cdot \frac{1}{1-(r+1)x^2-(r+1)x^4+x^6}. \end{aligned}$$

□

Corollary 18. *The aeration of $b(n;r)$ is the image of $a(n;r)$ under the Riordan array $\left(\frac{1}{1+x+x^3-x^4}, x\right)$.*

We now note that the Riordan array $\left(\frac{1}{1+x+x^3-x^4}, x\right)$ is the sequence array for the sequence with general term

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-j} F(n-2j+1).$$

Hence we have

Corollary 19.

$$b(n;r) = \sum_{k=0}^{2n} \sum_{j=0}^{\lfloor \frac{2n-k}{2} \rfloor} (-1)^{2n-k-j} F(2n-k-2j+1) a(k;r). \tag{20}$$

7 A Chebyshev product result

In this section, we observe that the polynomials

$$w(n;r) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} U_{n-2k}\left(\frac{r+2}{2}\right)$$

exhibit a multiplicative property. To this end, we let $U_n^d(x)$ represent a "doubled" Chebyshev polynomial, defined by

$$U_n^d(x) = U_{\frac{n}{2}}(x)\frac{1+(-1)^n}{2} + U_{\frac{n-1}{2}}(x)\frac{1-(-1)^n}{2}.$$

Clearly,

$$U_n^d(x) = \sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor - k}{k}(-1)^k(2x)^{\lfloor \frac{n}{2} \rfloor - 2k}.$$

The generating function for the polynomials $U_n^d(x)$ is then given by

$$\frac{1+y}{1-2xy^2+y^4}.$$

The generating function of $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} U_{n-2k}(x)$ is similarly given by

$$\frac{1}{(1-y^2)(1-2x+y^2)} = \frac{1}{1-2xy+2xy^3-y^4}.$$

Theorem 20. *We have*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} U_{n-2k}(x) = U_n^d(x)U_{n+1}^d(x). \tag{21}$$

Proof. The proof follows the same method as the proof of Theorem 5. For completeness, we give the details. We let

$$\frac{1+y}{1-2xy^2+y^4} = A(y)$$

so that

$$A(y) - 2xy^2A(y) + y^4A(y) = 1 + y.$$

Extracting the coefficient of y^{n+4}, we obtain

$$[y^{n+4}]A(y) - 2x[y^{n+2}]A(y) + [y^n]A(y) = [y^{n+4}](1+y)$$

$$A_{n+4} - 2xA_{n+2} + A_n = 0. \tag{22}$$

By direct inspection, the first terms of the sequence $A_n = a(n;r)$ are

$$1, 1, 2x, 2x, 4x^2-1, 4x^2-1, \ldots.$$

We now set $B_n = A_nA_{n+1}$ and $T_n = A_nA_{n+3}$, and try to determine the corresponding generating functions. From recurrence (22), we have

$$\begin{aligned} A_{n+4} &= 2xA_{n+2} - A_n \\ A_{n+5} &= 2xA_{n+3} - A_{n+1}. \end{aligned}$$

Multiplying, we obtain

$$A_{n+4}A_{n+5} = 4x^2A_{n+2}A_{n+3} - 2xA_{n+1}A_{n+2} - 2xA_nA_{n+3} + A_nA_{n+1}.$$

This can be written as:

$$B_{n+4} = 4x^2B_{n+2} - 2xB_{n+1} - 2xT_n + B_n. \tag{23}$$

Now multiplying equation (22) by A_{n+3}, we obtain

$$A_{n+3}A_{n+4} = 2xA_{n+2}A_{n+3} - A_nA_{n+3},$$

or

$$B_{n+3} = 2xB_{n+2} - T_n. \tag{24}$$

We can now use the method of coefficients to pass from the recurrences to the corresponding generating functions $B(y)$ and $T(y)$. In order to do so, we note that the coefficients of the sequence generated by $B(y)$ begin:

$$1, 2x, 4x^2, 2x(4x^2-1), \ldots.$$

Equation (23) becomes

$$\begin{aligned} &\frac{B(y) - 1 - 2xy - 4x^2y^2 - 2x(4x^2-1)y^3}{y^4} \\ &= 4x^2\frac{B(y) - 1 - 2xy}{y^2} - 2x\frac{B(y)-1}{y} - 2xT(y) + B(y) \end{aligned} \tag{25}$$

while Equation (24) becomes:

$$\frac{B(y) - 1 - 2xy - 4x^2y^2}{y^3} = 2x\frac{B(y) - 1 - 2xy}{y^2} - T(y). \tag{26}$$

We now have two linear equations for the two unknowns $B(y)$ and $T(y)$. Solving, we find that

$$B(y) = \frac{1}{(1-y^2)(1-2xy+y^2)}$$

as required. □

We note that we also obtain

$$T(y) = \frac{2x - y}{(1-y^2)(1-2xy+y^2)}.$$

This theorem then implies that

$$w(n;r)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} U_{n-2k}\left(\frac{r+2}{2}\right)=U_n^d\left(\frac{r+2}{2}\right)U_{n+1}^d\left(\frac{r+2}{2}\right). \tag{27}$$

An immediate consequence of this result is that we have

$$\begin{aligned}
b(n;r) &= \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} U_{n-2k}\left(\frac{r+2}{2}\right)-\sum_{k=0}^{\lfloor \frac{n-1}{2}\rfloor} U_{n-2k-1}\left(\frac{r+2}{2}\right)\\
&= U_n^d\left(\frac{r+2}{2}\right)U_{n+1}^d\left(\frac{r+2}{2}\right)-U_{n-1}^d\left(\frac{r+2}{2}\right)U_n^d\left(\frac{r+2}{2}\right)\\
&= U_n^d\left(\frac{r+2}{2}\right)\left(U_{n+1}^d\left(\frac{r+2}{2}\right)-U_{n-1}^d\left(\frac{r+2}{2}\right)\right)
\end{aligned}$$

This exhibits $b(n;r)$ as the product of a Chebyshev related polynomial and its first difference.

8 Additional Riordan array factorizations

We note that

$$\frac{1+x-x^2}{1-rx^2+x^4}=\left(\frac{1+x-x^2}{1+x^4},\frac{x^2}{1+x^4}\right)\cdot\frac{1}{1-rx}$$

and hence

$$\frac{1+x-x^2}{1-(r+2)x^2+x^4}=\left(\frac{1+x-x^2}{1+x^4},\frac{x^2}{1+x^4}\right)\left(\frac{1}{1-x},\frac{x}{1-x}\right)^2\cdot\frac{1}{1-rx}$$

Thus

$$\left(\frac{1+x-x^2}{(1-x^2)^2},\frac{x^2}{(1-x^2)^2}\right)=\left(\frac{1+x-x^2}{1+x^4},\frac{x^2}{1+x^4}\right)\left(\frac{1}{1-x},\frac{x}{1-x}\right)^2.$$

Here, $\left(\frac{1}{1-x},\frac{x}{1-x}\right)$ represents the standard binomial transformation. We also have

$$\begin{aligned}
(1+x+x^3-x^4,x^2)\cdot\left(\frac{1-x}{1-x^4},\frac{x}{1+x^2}\right) &= \left(\frac{1+x+x^3-x^4}{1+x^2+x^4+x^6},\frac{x^2}{1+x^4}\right)\\
&= \left(\frac{1+x-x^2}{1+x^4},\frac{x^2}{1+x^4}\right).
\end{aligned}$$

A consequence of this last factorization is the fact that

$$(1+x+x^3-x^4,x^2)\cdot\frac{1}{1-(r-1)x-(r-1)x^2+x^3}=\frac{1+x-x^2}{1-rx^2+x^4}.$$

9 A left inverse

We now seek a left inverse [1] for the array $(1+x+x^3-x^4, x^2)$. To this end we let $t=\sqrt{x}$. Then

$$\begin{aligned}
[x^n]\frac{\sqrt{x}^k}{1+\sqrt{x}+\sqrt{x}^3-\sqrt{x}^4} &= [t^{2n}]\frac{t^k}{1+t+t^3-t^4} \\
&= [t^{2n-k}]\frac{1}{1+t+t^3-t^4} \\
&= [t^{2n-k}]\sum_{j=0}^{\infty} u_j x^j \\
&= u_{2n-k}
\end{aligned}$$

where

$$u_n = (-1)^n F(\lfloor \frac{n+2}{2} \rfloor) F(\lfloor \frac{n+3}{2} \rfloor).$$

We deduce from this that

$$\begin{aligned}
b(n;r) &= \sum_{k=0}^{2n} (-1)^k F(\lfloor \frac{2n-k+2}{2} \rfloor) F(\lfloor \frac{2n-k+3}{2} \rfloor) \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} r^{\lfloor \frac{k-2j}{2} \rfloor} & (28) \\
&= \sum_{k=0}^{2n} (-1)^k F(n+1-\lfloor \frac{k}{2} \rfloor) F(n+1-\bmod(k,2)-\lfloor \frac{k}{2} \rfloor) a(k;r). & (29)
\end{aligned}$$

An alternative expression is given by

$$b(n;r) = \sum_{k=0}^{2n} \sum_{j=0}^{\frac{2n-k}{3}} \sum_{m=0}^{j} \binom{2n-k-2j-m}{j} \binom{j}{m} a(k;r). \qquad (30)$$

In matrix terms, we have, taking $a(n;2)$ (A002530), for example,

$$\left(\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
4 & -2 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
9 & -6 & 4 & -2 & 1 & -1 & 1 & 0 & 0 & 0 & 0\ldots \\
25 & -15 & 9 & -6 & 4 & -2 & 1 & -1 & 1 & 0 & 0\ldots \\
64 & -40 & 25 & -15 & 9 & -6 & 4 & -2 & 1 & -1 & 1\ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)
\left(\begin{array}{c}
1 \\ 1 \\ 3 \\ 4 \\ 11 \\ 15 \\ 41 \\ 56 \\ 153 \\ 209 \\ 571 \\ \vdots
\end{array}\right)
=
\left(\begin{array}{c}
1 \\ 3 \\ 12 \\ 44 \\ 165 \\ 615 \\ \vdots
\end{array}\right).$$

10 The matrix $\left(\frac{1-x}{1-x^4}, \frac{x}{1+x^2}\right)$ and its inverse

We recall that

$$\left(\frac{1-x}{1-x^4}, \frac{x}{1+x^2}\right) \cdot \frac{1}{1-(r+2)x} = \frac{1}{1-(r+1)x-(r+1)x^2+x^3}.$$

We have

$$\left(\frac{1-x}{1-x^4}, \frac{x}{1+x^2}\right) = \left(\frac{1}{1+x}, x\right) \cdot \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right),$$

where the last matrix is closely linked to the Chebyshev polynomials of the second kind. The matrix $\left(\frac{1-x}{1-x^4}, \frac{x}{1+x^2}\right)$ is A165620. We let $T_{n,k}$ denote the general element of $\left(\frac{1-x}{1-x^4}, \frac{x}{1+x^2}\right)$. The general term of $\left(\frac{1}{1+x}, x\right)$ is $[k \le n](-1)^{n-k}$ while that of $\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)$ is given by

$$(-1)^{\frac{n-k}{2}} \frac{(1+(-1)^{n-k}}{2} \binom{\frac{n+k}{2}}{k}.$$

Thus we have

$$T_{n,k} = \sum_{j=0..n} (-1)^{n-j} (-1)^{\frac{j-k}{2}} \frac{1+(-1)^{j-k}}{2} \binom{\frac{j+k}{2}}{k}.$$

We can then express $b(n;r)$ as

$$\begin{aligned} b(n;r) &= \sum_{k=0}^{n} T_{n,k}(r+2)^k \\ &= \sum_{k=0}^{n} T_{n,k} \sum_{j=0}^{k} \binom{k}{j} r^j 2^{k-j} \\ &= \sum_{k=0}^{n} \sum_{l=0}^{n} (-1)^{n-l} (-1)^{\frac{l-k}{2}} \frac{1+(-1)^{l-k}}{2} \binom{\frac{l+k}{2}}{k} \sum_{j=0}^{k} \binom{k}{j} r^j 2^{k-j}. \end{aligned}$$

This could also have been deduced from the fact that

$$\left(\frac{1-x}{1-x^4}, \frac{x}{1+x^2}\right) = \left(\frac{1}{(1-x)(1-x^2)}, \frac{x}{(1-x)^2}\right) \cdot \mathbf{B}^{-2}.$$

The row sums of $\left(\frac{1-x}{1-x^4}, \frac{x}{1+x^2}\right)$ have g.f. $\frac{1}{1+x^3}$ while the diagonal sums are the sequence $(-1)^n$. The inverse of this matrix is given by

$$\begin{aligned} (1+xc(x^2)+x^2c(x^2)^2+x^3c(x^2)^3, xc(x^2)) &= \left(\frac{1+x-2x^2-(1+x)\sqrt{1-4x^2}}{2x^3}, xc(x^2)\right) \\ &= \left(\frac{c(x^2)+xc(x^2)-1}{x}, xc(x^2)\right) \\ &= (c(x^2)(1+xc(x^2)), xc(x^2)) \\ &= (1+xc(x^2), x) \cdot (c(x^2), xc(x^2)). \end{aligned}$$

This matrix (A165621) begins

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 1 & 0 & 0 & \dots \\ 2 & 3 & 3 & 1 & 1 & 0 & \dots \\ 5 & 5 & 4 & 4 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the first column is the sequence

$$1, 1, 1, 2, 2, 5, 5, 14, 14, 42, \dots$$

of doubled Catalan numbers, which has moment representation

$$\frac{1}{2\pi} \int_{-2}^{2} x^n (1+x) \sqrt{4-x^2}\, dx$$

and Hankel transform [3]

$$1, 0, -1, -1, 0, 1, 1, 0, 0, -1, 1, \dots$$

A number of interesting observations may be made about this matrix. Since $g(x) = c(x^2)$, the A-sequence of this matrix has generating function $A(y) = 1 + y^2$. This then implies that

$$M_{n+1,k+1} = M_{n,k} + M_{n,k+2}.$$

The row sums of this matrix are the central binomial coefficients $\binom{n+1}{\lfloor \frac{n+1}{2} \rfloor}$. The diagonal sums are A026008. The transform of the sequence $1, 0, 1, 0, 1, 0, \dots$ by this matrix is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, A001405. The transform of $(-1)^n$ is the aerated Catalan numbers $1, 0, 1, 0, 2, 0, \dots$, while the transform of $(-2)^n$ is a sequence whose Hankel transform is $F(2n+1)$. In general, the transform of the sequence with general term r^n will have Hankel transform with g.f. $\frac{1-x}{1+(r-1)x+x^2}$ and general term

$$\sum_{k=0}^{n} \binom{2n-k}{k} (-(r+1))^{n-k}.$$

It is interesting to take the Hankel transform of the columns of this matrix. We obtain a matrix which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 1 & 0 & 0 & 0 & \dots \\ -1 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(where we have multiplied the columns by $(-1)^{\binom{k+1}{2}}$).

11 Appendix

In this section, we indicate a proof that the general term of the generalized Riordan array

$$\left(\frac{1+x-x^2}{(1-x^2)^2}, \frac{x^2}{(1-x^2)^2}\right)$$

is given by

$$\binom{\lfloor\frac{n+2k+1}{2}\rfloor}{\lfloor\frac{n-2k}{2}\rfloor}.$$

We note first that

$$\left(\frac{1+x-x^2}{(1-x^2)^2}, \frac{x^2}{(1-x^2)^2}\right) = (1+x-x^2, x)\left(\frac{1}{(1-x^2)^2}, \frac{x^2}{(1-x^2)^2}\right).$$

Now the Riordan array

$$\left(\frac{1}{(1-x)^2}, \frac{x}{(1-x)^2}\right),$$

which is the unsigned version of A053122, has general term

$$\binom{n+k+1}{2k+1}.$$

Thus the matrix

$$\left(\frac{1}{(1-x^2)^2}, \frac{x}{(1-x^2)^2}\right),$$

which represents an aeration of the matrix with general term $\binom{n+k+1}{2k+1}$, has general term

$$\binom{\frac{n+3k+2}{2}}{2k+1}\frac{1+(-1)^{n-k}}{2}.$$

Considering now the generalized Riordan matrix

$$\left(\frac{1}{(1-x)^2}, \frac{x^2}{(1-x)^2}\right)$$

is to effect the transformation $n \to n-k$ in the above formula. We thus obtain that the general element of the generalized Riordan array

$$\left(\frac{1}{(1-x)^2}, \frac{x^2}{(1-x)^2}\right)$$

is given by

$$\binom{\frac{n+2k+2}{2}}{2k+1}\frac{1+(-1)^{n-2k}}{2} = \binom{\frac{n}{2}+k+1}{2k+1}\frac{1+(-1)^n}{2}.$$

Thus the general term of

$$\left(\frac{1+x-x^2}{(1-x^2)^2}, \frac{x^2}{(1-x^2)^2}\right) = (1+x-x^2, x)\left(\frac{1}{(1-x^2)^2}, \frac{x^2}{(1-x^2)^2}\right)$$

is given by the sum

$$\binom{\frac{n}{2}+k+1}{2k+1}\frac{1+(-1)^n}{2} + \binom{\frac{n-1}{2}+k+1}{2k+1}\frac{1+(-1)^{n-1}}{2} - \binom{\frac{n-2}{2}+k+1}{2k+1}\frac{1+(-1)^{n-2}}{2}$$

or

$$\binom{\frac{n}{2}+k+1}{2k+1}\frac{1+(-1)^n}{2} + \binom{\frac{n}{2}+k+\frac{1}{2}}{2k+1}\frac{1-(-1)^n}{2} - \binom{\frac{n}{2}+k}{2k+1}\frac{1+(-1)^n}{2}$$

which simplifies to

$$\frac{2(2k+1)}{n-2k}\binom{\frac{n}{2}+k+1}{2k+1}\frac{1+(-1)^n}{2} + \binom{\frac{n}{2}+k+\frac{1}{2}}{2k+1}\frac{1-(-1)^n}{2}.$$

By considering alternately the cases $n = 2m$ and $n = 2m+1$ we can establish that this last term coincides with

$$\binom{\lfloor\frac{n+2k+1}{2}\rfloor}{\lfloor\frac{n-2k}{2}\rfloor}.$$

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