Meixner-type results for Riordan arrays and associated integer sequences

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Abstract

We determine which (ordinary) Riordan arrays are the coefficient arrays of a family of orthogonal polynomials. In so doing, we are led to introduce a family of polynomials, which includes the Boubaker polynomials, and a scaled version of the Chebyshev poynomials, using the techniques of Riordan arrays. We classify these polynomials in terms of the Chebyshev polynomials of the first and second kinds. We also examine the Hankel transforms of sequences associated to the inverse of the polynomial coefficient arrays, including the associated moment sequences.

1 Introduction

To each Riordan array $(A(t), B(t))$ we can associate a family of polynomials [19] by

$$\sum_{n=0}^{\infty} p_n(x)t^n = (A(t), B(t)) \cdot \frac{1}{1-xt} = \frac{A(t)}{1-xB(t)}.$$

The question can then be asked as to what conditions must be satisfied by $A(t)$ and $B(t)$ in order to ensure that $(p_n(x))_{n\geq 0}$ be a family of orthogonal polynomials. This can be considered to be a Meixner-type question [22], where the original Meixner result is related to Sheffer sequences (i.e., to exponential generating functions, rather than ordinary generating functions):

$$\sum_{n=0}^{\infty} p_n(x)t^n = A(t)\exp(xB(t)).$$

In providing an answer to this question, we shall introduce a two-parameter family of orthogonal polynomials using Riordan arrays. These polynomials are inspired by the well-known

Chebyshev polynomials [25], and the more recently introduced so-called Boubaker polynomials [2, 14, 16]. We shall classify these polynomials in terms of the Chebyshev polynomials of the first and second kinds, and we shall also examine properties of sequences related to the inverses of the coefficient arrays of the polynomials under study. While partly expository in nature, the note assumes a certain familiarity with integer sequences, generating functions, orthogonal polynomials [4, 10, 31], Riordan arrays [26, 30], production matrices [8, 24], and the integer Hankel transform [1, 6, 17]. Many interesting examples of sequences and Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences (OEIS), [28, 29]. Sequences are frequently referred to by their OEIS number. For instance, the binomial matrix $\mathbf{B}$ ("Pascal's triangle") is A007318.

The plan of the paper is as follows:

1. This Introduction
2. Preliminaries on integer sequences and Riordan arrays
3. Orthogonal polynomials and Riordan arrays
4. Riordan arrays, production matrices and orthogonal polynomials
5. Chebyshev polynomials and Riordan arrays
6. The Boubaker polynomials
7. The family of Chebyshev-Boubaker polynomials
8. The inverse matrix $\mathfrak{B}^{-1}$
9. A curious relation
10. Acknowledgements

2 Preliminaries on integer sequences and Riordan arrays

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is called the *ordinary generating function* or g.f. of the sequence. a_n is thus the coefficient of x^n in this series. We denote this by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ is the n-th Catalan number A000108. We use the notation $0^n = [x^n]1$ for the sequence $1, 0, 0, 0, \ldots,$ A000007. Thus $0^n = [n = 0] = \delta_{n,0} = \binom{0}{n}$. Here, we have used the Iverson bracket notation [11], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false.

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)$ such that $f(\bar{f}(x)) = x$. We sometimes write $\bar{f} = \text{Rev} f$.

For a lower triangular matrix $(a_{n,k})_{n,k\geq 0}$ the row sums give the sequence with general term $\sum_{k=0}^{n} a_{n,k}$ while the diagonal sums form the sequence with general term

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-k,k}.$$

The *Riordan group* [26, 30], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1 x + g_2 x^2 + \dots$ and $f(x) = f_1 x + f_2 x^2 + \dots$ where $g_0 \neq 0$ and $f_1 \neq 0$ [30]. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g, f)(h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

A Riordan array of the form $(g(x), x)$, where $g(x)$ is the generating function of the sequence a_n, is called the *sequence array* of the sequence a_n. Its general term is a_{n-k}. Such arrays are also called *Appell* arrays as they form the elements of the Appell subgroup.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \dots)'$ is an integer sequence with ordinary generating function $\mathcal{A}\,(x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbb{Z}^{\mathbb{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbb{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \mapsto (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

In [18, 19] the notation $T(f|g)$ is used to denote the Riordan array

$$T(f|g) = \left(\frac{f(x)}{g(x)}, \frac{x}{g(x)} \right).$$

Example 1. The so-called *binomial matrix* $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. Thus

$$\mathbf{B} = T(1|1 - x).$$

This matrix has general element $\binom{n}{k}$, and hence as an array coincides with Pascal's triangle. More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k} m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

Example 2. If a_n has generating function $g(x)$, then the generating function of the sequence

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-2k}$$

is equal to

$$\frac{g(x)}{1-x^2} = \left(\frac{1}{1-x^2}, x\right) \cdot g(x),$$

while the generating function of the sequence

$$d_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a_{n-2k}$$

is equal to

$$\frac{1}{1-x^2} g\left(\frac{x}{1-x^2}\right) = \left(\frac{1}{1-x^2}, \frac{x}{1-x^2}\right) \cdot g(x).$$

The row sums of the matrix (g, f) have generating function

$$(g, f) \cdot \frac{1}{1-x} = \frac{g(x)}{1-f(x)}$$

while the diagonal sums of (g, f) (sums of left-to-right diagonals in the North East direction) have generating function $g(x)/(1 - xf(x))$. These coincide with the row sums of the "generalized" Riordan array (g, xf):

$$(g, xf) \cdot \frac{1}{1-x} = \frac{g(x)}{1-xf(x)}.$$

For instance the Fibonacci numbers F_{n+1} are the diagonal sums of the binomial matrix $\mathbf{B}$ given by $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$:

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & 0 & \dots \\ 1 & 5 & 10 & 10 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

while they are the row sums of the "generalized" or "stretched" (using the nomenclature of [5]) Riordan array $\left(\frac{1}{1-x}, \frac{x^2}{1-x}\right)$:

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 4 & 3 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

It is often the case that we work with "generalized" Riordan arrays, where we relax some of the defining conditions above. Thus for instance [5] discusses the notion of the "stretched" Riordan array. In this note, we shall encounter "vertically stretched" arrays of the form (g, h) where now $f_0 = f_1 = 0$ with $f_2 \neq 0$. Such arrays are not invertible, but we may explore their left inversion. In this context, standard Riordan arrays as described above are called "proper" Riordan arrays. We note for instance that for any proper Riordan array (g, f), its diagonal sums are just the row sums of the vertically stretched array (g, xf) and hence have g.f. $g/(1 - xf)$.

Each Riordan array $(g(x), f(x))$ has bi-variate generating function given by

$$\frac{g(x)}{1 - yf(x)}.$$

For instance, the binomial matrix $\mathbf{B}$ has generating function

$$\frac{\frac{1}{1-x}}{1 - y\frac{x}{1-x}} = \frac{1}{1 - x(1 + y)}.$$

For a sequence $a_0, a_1, a_2, \ldots$ with g.f. $g(x)$, the "aeration" of the sequence is the sequence $a_0, 0, a_1, 0, a_2, \ldots$ with interpolated zeros. Its g.f. is $g(x^2)$.

The aeration of a (lower-triangular) matrix $\mathbf{M}$ with general term $m_{i,j}$ is the matrix whose general term is given by

$$m^r_{\frac{i+j}{2}, \frac{i-j}{2}} \frac{1 + (-1)^{i-j}}{2},$$

where $m^r_{i,j}$ is the i, j-th element of the reversal of $\mathbf{M}$:

$$m^r_{i,j} = m_{i,i-j}.$$

In the case of a Riordan array (or indeed any lower triangular array), the row sums of the aeration are equal to the diagonal sums of the reversal of the original matrix.

Example 3. The Riordan array $(c(x^2), xc(x^2))$ is the aeration of $(c(x), xc(x))$ A033184. Here

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is the g.f. of the Catalan numbers. Indeed, the reversal of $(c(x), xc(x))$ is the matrix with general element

$$[k \leq n + 1]\binom{n + k}{k}\frac{n - k + 1}{n + 1},$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 2 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 5 & 5 & 0 & 0 & \cdots \\ 1 & 4 & 9 & 14 & 14 & 0 & \cdots \\ 1 & 5 & 14 & 28 & 42 & 42 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A009766. Then $(c(x^2), xc(x^2))$ has general element

$$\binom{n+1}{\frac{n-k}{2}}\frac{k+1}{n+1}\frac{(1+(-1)^{n-k}}{2},$$

and begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 2 & 0 & 3 & 0 & 1 & 0 & \cdots \\ 0 & 5 & 0 & 4 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A053121. We have

$$(c(x^2), xc(x^2)) = \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}.$$

We note that the diagonal sums of the reverse of $(c(x), xc(x))$ coincide with the row sums of $(c(x^2), xc(x^2))$, and are equal to the central binomial coefficients $\binom{n}{\lfloor\frac{n}{2}\rfloor}$ A001405.

An important feature of Riordan arrays is that they have a number of sequence characterizations [3, 15]. The simplest of these is as follows.

Proposition 4. *[15] Let $D = [d_{n,k}]$ be an infinite triangular matrix. Then D is a Riordan array if and only if there exist two sequences $A = [a_0, a_1, a_2, \ldots]$ and $Z = [z_0, z_1, z_2, \ldots]$ with $a_0 \neq 0$ such that*

- $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}, \quad (k, n = 0, 1, \ldots)$
- $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}, \quad (n = 0, 1, \ldots).$

The coefficients $a_0, a_1, a_2, \ldots$ and $z_0, z_1, z_2, \ldots$ are called the A-sequence and the Z-sequence of the Riordan array $D = (g(x), f(x))$, respectively. Letting $A(x)$ and $Z(x)$ denote the generating functions of these sequences, respectively, we have [20] that

$$\frac{f(x)}{x} = A(f(x)), \quad g(x) = \frac{d_{0,0}}{1 - xZ(f(x))}.$$

We therefore deduce that

$$A(x) = \frac{x}{\bar{f}(x)},$$

and

$$Z(x) = \frac{1}{\bar{f}(x)}\left[1 - \frac{d_{0,0}}{g(\bar{f}(x))}\right].$$

A consequence of this is the following result, which was originally established [19] by Luzón:

Lemma 5. *Let $D = (g, f)$ be a Riordan array, whose A-sequence, respectively Z-sequence have generating functions $A(x)$ and $Z(x)$. Then*

$$D^{-1} = \left(\frac{A - xZ}{d_{0,0}A}, \frac{x}{A}\right).$$

3 Orthogonal polynomials and Riordan arrays

By an *orthogonal polynomial sequence* $(p_n(x))_{n\geq 0}$ we shall understand [4, 10] an infinite sequence of polynomials $p_n(x)$, $n \geq 0$, of degree n, with real coefficients (often integer coefficients) that are mutually orthogonal on an interval $[x_0, x_1]$ (where $x_0 = -\infty$ is allowed, as well as $x_1 = \infty$), with respect to a weight function $w : [x_0, x_1] \to \mathbb{R}$:

$$\int_{x_0}^{x_1} p_n(x)p_m(x)w(x)dx = \delta_{nm}\sqrt{h_n h_m},$$

where

$$\int_{x_0}^{x_1} p_n^2(x)w(x)dx = h_n.$$

We assume that w is strictly positive on the interval (x_0, x_1). Every such sequence obeys a so-called "three-term recurrence" :

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x)$$

for coefficients a_n, b_n and c_n that depend on n but not x. We note that if

$$p_j(x) = k_j x^j + k'_j x^{j-1} + \dots \qquad j = 0, 1, \dots$$

then

$$a_n = \frac{k_{n+1}}{k_n}, \qquad b_n = a_n\left(\frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n}\right), \qquad c_n = a_n\left(\frac{k_{n-1}h_n}{k_n h_{n-1}}\right),$$

where

$$h_i = \int_{x_0}^{x_1} p_i(x)^2 w(x)\,dx.$$

Since the degree of $p_n(x)$ is n, the coefficient array of the polynomials is a lower triangular (infinite) matrix. In the case of monic orthogonal polynomials the diagonal elements of this array will all be 1. In this case, we can write the three-term recurrence as

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0.$$

The *moments* associated to the orthogonal polynomial sequence are the numbers

$$\mu_n = \int_{x_0}^{x_1} x^n w(x)dx.$$

We can find $p_n(x)$, α_n and β_n from a knowledge of these moments. To do this, we let Δ_n be the Hankel determinant $|\mu_{i+j}|_{i,j\geq 0}^n$ and $\Delta_{n,x}$ be the same determinant, but with the last row equal to $1, x, x^2, \ldots$. Then

$$p_n(x) = \frac{\Delta_{n,x}}{\Delta_{n-1}}.$$

More generally, we let $H\begin{pmatrix} u_1 & \ldots & u_k \\ v_1 & \ldots & v_k \end{pmatrix}$ be the determinant of Hankel type with (i,j)-th term $\mu_{u_i+v_j}$. Let

$$\Delta_n = H\begin{pmatrix} 0 & 1 & \ldots & n \\ 0 & 1 & \ldots & n \end{pmatrix}, \qquad \Delta' = H\begin{pmatrix} 0 & 1 & \ldots & n-1 & n \\ 0 & 1 & \ldots & n-1 & n+1 \end{pmatrix}.$$

Then we have

$$\alpha_n = \frac{\Delta'_n}{\Delta_n} - \frac{\Delta'_{n-1}}{\Delta_{n-1}}, \qquad \beta_n = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}.$$

Of importance to this study are the following results (the first is the well-known "Favard's Theorem"), which we essentially reproduce from [13].

Theorem 6. *[13] (Cf. [32], Théorème* 9 *on p.I-4, or [33], Theorem* 50.1*). Let $(p_n(x))_{n\geq 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree $n = 0, 1, \ldots$ Then the sequence $(p_n(x))$ is (formally) orthogonal if and only if there exist sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 1}$ with $\beta_n \neq 0$ for all $n \geq 1$, such that the three-term recurrence*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n(x), \quad \textit{for} \quad n \geq 1,$$

holds, with initial conditions $p_0(x) = 1$ and $p_1(x) = x - \alpha_0$.

Theorem 7. *[13] (Cf. [32], Proposition 1, (7), on p. V-5, or [33], Theorem* 51.1*). Let $(p_n(x))_{n\geq 0}$ be a sequence of monic polynomials, which is orthogonal with respect to some functional L. Let*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n(x), \quad \textit{for} \quad n \geq 1,$$

be the corresponding three-term recurrence which is guarenteed by Favard's theorem. Then the generating function

$$g(x) = \sum_{k=0}^{\infty} \mu_k x^k$$

for the moments $\mu_k = L(x^k)$ satisfies

$$g(x) = \cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}.$$

Given a family of monic orthogonal polynomials

$$p_{n+1}(x) = (x-\alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0,$$

we can write

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k.$$

Then we have

$$\sum_{k=0}^{n+1} a_{n+1,k} x^k = (x-\alpha_n) \sum_{k=0}^{n} a_{n,k} x^k - \beta_n \sum_{k=0}^{n-1} a_{n-1,k} x^k$$

from which we deduce

$$a_{n+1,0} = -\alpha_n a_{n,0} - \beta_n a_{n-1,0} \tag{1}$$

and

$$a_{n+1,k} = a_{n,k-1} - \alpha_n a_{n,k} - \beta_n a_{n-1,k} \tag{2}$$

The question immediately arises as to the conditions under which a Riordan array (g, f) can be the coefficient array of a family of orthogonal polynomials. A partial answer is given by the following proposition.

Proposition 8. *Every Riordan array of the form*

$$\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$$

is the coefficient array of a family of monic orthogonal polynomials.

Proof. By [12], the array $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ has a C-sequence $C(x) = \sum_{n\geq 0} c_n x^n$ given by

$$\frac{x}{1+rx+sx^2} = \frac{x}{1-xC(x)},$$

and thus

$$C(x) = -r - sx.$$

Thus the Riordan array $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ is determined by the fact that

$$a_{n+1,k} = a_{n,k-1} + \sum_{i\geq 0} c_i d_{n-i,k} \quad \text{for } n, k = 0, 1, 2, \ldots$$

where $a_{n,-1} = 0$. In the case of $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ we have

$$a_{n+1,k} = a_{n,k-1} - r a_{n,k} - s a_{n-1,k}.$$

Working backwards, this now ensures that

$$p_{n+1}(x) = (x-r)p_n(x) - s p_{n-1}(x),$$

where $p_n(x) = \sum_{k=0}^{n} a_{n,k} x^n$. □

We note that in this case the three-term recurrence coefficients α_n and β_n are constants.

We can strengthen this result as follows.

Proposition 9. *Every Riordan array of the form*

$$\left(\frac{1-\lambda x-\mu x^2}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$$

is the coefficient array of a family of monic orthogonal polynomials.

Proof. We have

$$\left(\frac{1-\lambda x-\mu x^2}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right) = (1-\lambda x-\mu x^2, x)\cdot\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right),$$

where $(1-\lambda x-\mu x^2, x)$ is the array with elements

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\lambda & 1 & 0 & 0 & 0 & 0 & \dots \\ -\mu & -\lambda & 1 & 0 & 0 & 0 & \dots \\ 0 & -\mu & -\lambda & 1 & 0 & 0 & \dots \\ 0 & 0 & -\mu & -\lambda & 1 & 0 & \dots \\ 0 & 0 & 0 & -\mu & -\lambda & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We write

$$B=(b_{n,k})=\left(\frac{1-\lambda x-\mu x^2}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right),$$

and

$$A=(a_{n,k})=\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right),$$

where

$$a_{n+1,k}=a_{n,k-1}-ra_{n,k}-sa_{n-1,k}. \quad (3)$$

We now assert that also,

$$b_{n+1,k}=b_{n,k-1}-rb_{n,k}-sb_{n-1,k}.$$

This follows since the fact that

$$B=(1-\lambda x-\mu x^2, x)\cdot A$$

tells us that

$$\begin{aligned} b_{n+1,k} &= a_{n+1,k}-\lambda a_{n,k}-\mu a_{n-1,k}, \\ b_{n,k-1} &= a_{n,k-1}-\lambda a_{n-1,k-1}-\mu a_{n-2,k-1}, \\ b_{n,k} &= a_{n,k}-\lambda a_{n-1,k}-\mu a_{n-2,k}, \\ b_{n-1,k} &= a_{n-1,k}-\lambda a_{n-2,k}-\mu a_{n-3,k}. \end{aligned}$$

Then using equation (3) and the equivalent equations for $a_{n,k}$ and $a_{n-1,k}$, we see that

$$b_{n+1,k} = b_{n,k-1} - rb_{n,k} - sb_{n-1,k}$$

as required. Noting that

$$p_0(x) = 1, \quad p_1(x) = x - r - \lambda, \quad p_2(x) = x^2 - (2r + \lambda)x + \lambda r - \mu + r^2 - s, \ldots,$$

we see that the family of orthogonal polynomials is defined by the α-sequence

$$\alpha_0 = r + \lambda, r, r, r, \ldots$$

and the β-sequence

$$\beta_1 = s + \mu, s, s, s, \ldots.$$

□

Proposition 10. *The elements in the left-most column of*

$$L = \left(\frac{1 - \lambda x - \mu x^2}{1 + rx + sx^2}, \frac{x}{1 + rx + sx^2} \right)^{-1}$$

are the moments corresponding to the family of orthogonal polynomials with coefficient array L^{-1}.

Proof. We let

$$(g, f) = \left(\frac{1 - \lambda x - \mu x^2}{1 + rx + sx^2}, \frac{x}{1 + rx + sx^2} \right).$$

Then

$$L = (g, f)^{-1} = \left(\frac{1}{g(\bar{f})}, \bar{f} \right).$$

Now $\bar{f}(x)$ is the solution to

$$\frac{u}{1 + rx + sx^2} = x,$$

thus

$$\bar{f}(x) = \frac{1 - sx - \sqrt{1 - 2sx + (s^2 - 4r)x^2}}{2rx}.$$

Then

$$\frac{1}{g(\bar{f}(x))} = \frac{1 + r\bar{f}(x) + s(\bar{f}(x))^2}{1 - \lambda \bar{f}(x) - \mu(\bar{f}(x))^2}.$$

Simplifying,we find that

$$\frac{1}{g(\bar{f}(x))} = \frac{2s}{(s + \mu)\sqrt{1 - 2rx + (r^2 - 4s)x^2} - (r(s - \mu) + 2s\lambda)x + s - \mu}.$$

We now consider the continued fraction

$$\tilde{g}(x) = \cfrac{1}{1-(r+\lambda)x - \cfrac{(s+\mu)x^2}{1-rx-\cfrac{sx^2}{1-rx-\cfrac{sx^2}{1-rx-\cdots}}}}.$$

This is equivalent to

$$\tilde{g}(x) = \frac{1}{1-(r+\lambda)x-(s+\mu)x^2h(x)},$$

where

$$h(x) = \frac{1}{1-rx-sx^2h(x)}.$$

Solving for $h(x)$ and subsequently for $\tilde{g}(x)$, we find that

$$\tilde{g}(x) = \frac{1}{g(\bar{f}(x))}.$$

□

We have in fact the following proposition (see the next section for information on the Chebyshev polynomials).

Proposition 11. *The Riordan array* $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ *is the coefficient array of the modified Chebyshev polynomials of the second kind given by*

$$P_n(x) = (\sqrt{s})^n U_n\left(\frac{x-r}{2\sqrt{s}}\right), \quad n = 0, 1, 2, \ldots$$

Proof. We have

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n.$$

Thus

$$\frac{1}{1-2\frac{x-r}{2\sqrt{s}}\sqrt{s}t+st^2} = \sum_{n=0}^{\infty} U_n\left(\frac{x-r}{2\sqrt{s}}\right)(\sqrt{s}t)^n.$$

Now

$$\begin{aligned}\frac{1}{1-2\frac{x-r}{2\sqrt{s}}\sqrt{s}t+st^2} &= \frac{1}{1-(x-r)t+st^2} \\ &= \left(\frac{1}{1+rt+st^2}, \frac{t}{1+rt+st^2}\right)\cdot\frac{1}{1-xt}.\end{aligned}$$

Thus

$$\left(\frac{1}{1+rt+st^2}, \frac{t}{1+rt+st^2}\right)\cdot\frac{1}{1-xt} = \sum_{n=0}^{\infty}(\sqrt{s})^n U_n\left(\frac{x-r}{2\sqrt{s}}\right)t^n$$

as required. □

For another perspective on this result, see [9].

Corollary 12. *The Riordan array* $\left(\frac{1-\lambda x-\mu x^2}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ *is the coefficient array of the generalized Chebyshev polynomials of the second kind given by*

$$Q_n(x) = (\sqrt{s})^n U_n\left(\frac{x-r}{2\sqrt{s}}\right) - \lambda(\sqrt{s})^{n-1} U_{n-1}\left(\frac{x-r}{2\sqrt{s}}\right) - \mu(\sqrt{s})^{n-2} U_{n-2}\left(\frac{x-r}{2\sqrt{s}}\right), \quad n = 0, 1, 2, \ldots$$

Proof. We have

$$U_n(x) = [x^n]\frac{1}{1-2xt+t^2}$$

By the method of coefficients [21] we then have

$$[x^n]\frac{t}{1-2xt+t^2} = [x^{n-1}]\frac{1}{1-2xt+t^2} = U_{n-1}(x)$$

and similarly

$$[x^n]\frac{t^2}{1-2xt+t^2} = [x^{n-2}]\frac{1}{1-2xt+t^2} = U_{n-2}(x).$$

□

A more complete answer to our original question can be found by considering the associated production matrix [7, 8] of a Riordan arrray, which we look at in the next section.

4 Riordan arrays, production matrices and orthogonal polynomials

The concept of a *production matrix* [7, 8] is a general one, but for this work we find it convenient to review it in the context of Riordan arrays. Thus let P be an infinite matrix (most often it will have integer entries). Letting $\mathbf{r}_0$ be the row vector

$$\mathbf{r}_0 = (1, 0, 0, 0, \ldots),$$

we define $\mathbf{r}_i = \mathbf{r}_{i-1}P$, $i \geq 1$. Stacking these rows leads to another infinite matrix which we denote by A_P. Then P is said to be the *production matrix* for A_P.
If we let

$$u^T = (1, 0, 0, 0, \ldots, 0, \ldots)$$

then we have

$$A_P = \begin{pmatrix} u^T \\ u^T P \\ u^T P^2 \\ \vdots \end{pmatrix}$$

and

$$DA_P = A_P P$$

where $D = (\delta_{i+1,j})_{i,j\geq 0}$ (where δ is the usual Kronecker symbol).

In [24, 27] P is called the Stieltjes matrix associated to A_P.
The sequence formed by the row sums of A_P often has combinatorial significance and is called the sequence associated to P. Its general term a_n is given by $a_n = u^T P^n e$ where

$$e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

In the context of Riordan arrays, the production matrix associated to a proper Riordan array takes on a special form :

Proposition 13. *[8] Let P be an infinite production matrix and let A_P be the matrix induced by P. Then A_P is an (ordinary) Riordan matrix if and only if P is of the form*

$$P = \begin{pmatrix} \xi_0 & \alpha_0 & 0 & 0 & 0 & 0 & \cdots \\ \xi_1 & \alpha_1 & \alpha_0 & 0 & 0 & 0 & \cdots \\ \xi_2 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \cdots \\ \xi_3 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \cdots \\ \xi_4 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \cdots \\ \xi_5 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Moreover, columns 0 and 1 of the matrix P are the Z- and A-sequences, respectively, of the Riordan array A_P.

Example 14. We calculate the production matrix of the Riordan array

$$D = (c(x), xc(x)).$$

We have

$$f(x) = xc(x) \Rightarrow \bar{f}(x) = x(1-x),$$

and hence

$$A(x) = \frac{x}{\bar{f}(x)} = \frac{x}{x(1-x)} = \frac{1}{1-x}.$$

Similarly,

$$\begin{aligned} Z(x) &= \frac{1}{\bar{f}(x)}\left[1 - \frac{d_{0,0}}{g(\bar{f}(x))}\right] \\ &= \frac{1}{x(1-x)}\left[1 - \frac{1}{c(x(1-x))}\right] \\ &= \frac{1}{x(1-x)}\left[1 - \frac{1}{\frac{1}{1-x}}\right] \\ &= \frac{1}{x(1-x)}\left[1-(1-x)\right] \\ &= \frac{1}{1-x}. \end{aligned}$$

Thus the production matrix of $D = (c(x), xc(x))$ is the matrix that begins

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 15. We calculate the production matrix of the Riordan array

$$(g, f) = (c(x^2), xc(x^2)) = \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}.$$

First, we have

$$f(x) = xc(x^2) \Rightarrow \bar{f}(x) = \frac{x}{1+x^2},$$

and hence

$$A(x) = \frac{x}{\bar{f}(x)} = 1 + x^2.$$

Next, since

$$\frac{1}{g(\bar{f}(x))} = \frac{1}{1+x^2},$$

we have

$$\begin{aligned} Z(x) &= \frac{1}{\bar{f}(x)}\left[1 - \frac{d_{0,0}}{g(\bar{f}(x))}\right] \\ &= \frac{1+x^2}{x}\left[1 - \frac{1}{1+x^2}\right] \\ &= \frac{1+x^2}{x}\left[\frac{1+x^2-1}{1+x^2}\right] \\ &= x. \end{aligned}$$

Hence the production matrix of $(c(x^2), xc(x^2))$ begins

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can generalize the last result to give the following important result.

Proposition 16. *The Riordan array L where*

$$L^{-1} = \left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)$$

has production matrix (Stieltjes matrix) given by

$$P = S_L = \begin{pmatrix} a+\lambda & 1 & 0 & 0 & 0 & 0 & \dots \\ b+\mu & a & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. We let

$$(g, f) = L = \left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}.$$

By definition of the inverse, we have

$$\bar{f}(x) = \frac{x}{1+ax+bx^2}$$

and hence

$$A(x) = \frac{x}{\bar{f}(x)} = 1+ax+bx^2.$$

Also by definition of the inverse, we have

$$\frac{1}{g(\bar{f}(x))} = \frac{1-\lambda x-\mu x^2}{1+ax+bx^2},$$

and hence

$$\begin{aligned} Z(x) &= \frac{1}{\bar{f}(x)}\left[1-\frac{d_{0,0}}{g(\bar{f}(x))}\right] \\ &= \frac{1+ax+bx^2}{x}\left[1-\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}\right] \\ &= \frac{1+ax+bx^2}{x}\left[1+ax+bx^2-1+\lambda x+\mu x^2\right] \\ &= (a+\lambda)+(b+\mu)x. \end{aligned}$$

□

Corollary 17. *The Riordan array*

$$L = \left(\frac{1+(a-a_1)x+(b-b_1)x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}$$

has production matrix that begins

$$P = S_L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ b_1 & a & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 18. We note that since

$$\begin{aligned} L^{-1} &= \left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right) \\ &= (1-\lambda x-\mu x^2, x) \cdot \left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right), \end{aligned}$$

we have

$$L = \left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1} = \left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1} \cdot \left(\frac{1}{1-\lambda x-\mu x^2}, x\right).$$

If we now let

$$L_1 = \left(\frac{1}{1+ax}, \frac{x}{1+ax}\right) \cdot L,$$

then (see [23]) we obtain that the Stieltjes matrix for L_1 is given by

$$S_{L_1} = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & \dots \\ b+\mu & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & b & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & b & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & b & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & b & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have in fact the following general result [23]:

Proposition 19. *[23] If $L = (g(x), f(x))$ is a Riordan array and $P = S_L$ is tridiagonal, then necessarily*

$$P = S_L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ b_1 & a & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$f(x) = Rev\frac{x}{1+ax+bx^2} \qquad \text{and} \qquad g(x) = \frac{1}{1-a_1x-b_1xf},$$

and vice-versa.

Of central importance to this note is the following result.

Proposition 20. *If $L = (g(x), f(x))$ is a Riordan array and $P = S_L$ is tridiagonal of the form*

$$P = S_L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ b_1 & a & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & a & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & a & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & a & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & a & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4}$$

then L^{-1} is the coefficient array of the family of orthogonal polynomials $p_n(x)$ where $p_0(x) = 1$, $p_1(x) = x - a_1$, and

$$p_{n+1}(x) = (x-a)p_n(x) - b_n p_{n-1}(x), \qquad n \geq 2,$$

where b_n is the sequence $0, b_1, b, b, b, \ldots$.

Proof. (We are indebted to an anonymous reviewer for the form of the proof that follows). The form of the matrix P in (4) is equivalent to saying that $A(x) = 1 + ax + bx^2$ and that $Z(x) = a_1 + b_1x$. Now Lemma 5 tells us that if (d, h) is a Riordan array with A and Z the corresponding A-sequence and Z-sequence, respectively, then

$$(d, h)^{-1} = \left(\frac{A - xZ}{d_{0,0}A}, \frac{x}{A}\right).$$

Note that by assumption, $d_{0,0} = 1$ here. Thus

$$L^{-1} = \left(\frac{1+(a-a_1)x+(b-b_1)x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right) = T(1+(a-a_1)x+(b-b_1)x^2|1+ax+bx^2).$$

Theorem 5 of [19] now yields the required form of the three-term recurrence for the associated polynomials with coefficient array L^{-1}. That these are orthogonal polynomials then follows by Favard's theorem. □

We note that the elements of the rows of L^{-1} can be identified with the coefficients of the characteristic polynomials of the successive principal sub-matrices of P.

Example 21. We consider the Riordan array

$$\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right).$$

Then the production matrix (Stieltjes matrix) of the inverse Riordan array $\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}$ left-multiplied by the k-th binomial array

$$\left(\frac{1}{1-kx}, \frac{x}{1-kx}\right) = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)^k$$

is given by

$$P = \begin{pmatrix} a+k & 1 & 0 & 0 & 0 & 0 & \dots \\ b & a+k & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a+k & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a+k & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a+k & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a+k & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and vice-versa. This follows since

$$\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right) \cdot \left(\frac{1}{1+kx}, \frac{x}{1+kx}\right) = \left(\frac{1}{1+(a+k)x+bx^2}, \frac{x}{1+(a+k)x+bx^2}\right).$$

In fact we have the more general result :

$$\left(\frac{1+\lambda x+\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right) \cdot \left(\frac{1}{1+kx}, \frac{x}{1+kx}\right) = \left(\frac{1+\lambda x+\mu x^2}{1+(a+k)x+bx^2}, \frac{x}{1+(a+k)x+bx^2}\right).$$

The inverse of this last matrix therefore has production array

$$\begin{pmatrix} a+k-\lambda & 1 & 0 & 0 & 0 & 0 & \dots \\ b-\mu & a+k & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a+k & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a+k & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a+k & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a+k & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that if $L = (g(x), f(x))$ is a Riordan array and $P = S_L$ is tridiagonal of the form given in Eq. (4), then the first column of L gives the moment sequence for the weight function associated to the orthogonal polynomials whose coefficient array is L^{-1}.

As pointed out by a referee (to whom we are indebted for this important observation), the main results of the last two sections may be summarized as follows:

Proposition 22. *Let $L = (d(x), h(x))$ be a Riordan array. Then the following are equivalent:*

1. *L is the coefficient array of a family of monic orthogonal polynomials*
2. *$d(x) = \frac{1-\lambda x-\mu x^2}{1+rx+sx^2}$ and $h(x) = \frac{x}{1+rx+sx^2}$ with $s \neq 0$.*

3. The production matrix of L^{-1} *is of the form*

$$P = S_{L^{-1}} = \begin{pmatrix} r_1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ s_1 & r & 1 & 0 & 0 & 0 & \ldots \\ 0 & s & r & 1 & 0 & 0 & \ldots \\ 0 & 0 & s & r & 1 & 0 & \ldots \\ 0 & 0 & 0 & s & r & 1 & \ldots \\ 0 & 0 & 0 & 0 & s & r & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $s \neq 0$.

4. The bivariate generating function of L *is of the form*

$$\frac{1 - \lambda x - \mu x^2}{1 + (r - t)x + sx^2}$$

with $s \neq 0$.

Under these circumstances, the elements of the left-most column of L^{-1} are the moments associated to the linear functional that defines the family of orthogonal polynomials.

5 Chebyshev polynomials and Riordan arrays

We begin this section by recalling some facts about the Chebyshev polynomials of the first and second kind. Thus the Chebyshev polynomials of the first kind, $T_n(x)$, are defined by

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{(-1)^k}{n-k} (2x)^{n-2k} \tag{5}$$

for $n > 0$, and $T_0(x) = 1$. Similarly, the Chebyshev polynomials of the second kind, $U_n(x)$, can be defined by

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-1)^k (2x)^{n-2k}, \tag{6}$$

or alternatively as

$$U_n(x) = \sum_{k=0}^{n} \binom{\frac{n+k}{2}}{k} (-1)^{\frac{n-k}{2}} \frac{1 + (-1)^{n-k}}{2} (2x)^k. \tag{7}$$

In terms of generating functions, we have

$$\sum_{n=0}^{\infty} T_n(x) t^n = \frac{1 - xt}{1 - 2xt + t^2},$$

while

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1-2xt+t^2}.$$

The Chebyshev polynomials of the second kind, $U_n(x)$, which begin

$$1, 2x, 4x^2-1, 8x^3-4x, 16x^4-12x^2+1, 32x^5-32x^3+6x, \ldots$$

have coefficient array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 4 & 0 & 0 & 0 & \ldots \\ 0 & -4 & 0 & 8 & 0 & 0 & \ldots \\ 1 & 0 & -12 & 0 & 16 & 0 & \ldots \\ 0 & 6 & 0 & -32 & 0 & 32 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \text{(A053117)}$$

This is the (generalized) Riordan array

$$\left(\frac{1}{1+x^2}, \frac{2x}{1+x^2}\right).$$

We note that the coefficient array of the modified Chebyshev polynomials $U_n(x/2)$ which begin

$$1, x, x^2-1, x^3-2x, x^4-3x^2+1, \ldots,$$

is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & -2 & 0 & 1 & 0 & 0 & \ldots \\ 1 & 0 & -3 & 0 & 1 & 0 & \ldots \\ 0 & 3 & 0 & -4 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \text{(A049310)}$$

This is the Riordan array

$$\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right).$$

The situation with the Chebyshev polynomials of the first kind is more complicated, since while the coefficient array of the polynomials $2T_n(x) - 0^n$, which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 0 & 0 & 0 & 0 & \ldots \\ -2 & 0 & 4 & 0 & 0 & 0 & \ldots \\ 0 & -6 & 0 & 8 & 0 & 0 & \ldots \\ 2 & 0 & -16 & 0 & 16 & 0 & \ldots \\ 0 & 10 & 0 & -40 & 0 & 32 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a (generalized) Riordan array, namely

$$\left(\frac{1-x^2}{1+x^2}, \frac{2x}{1+x^2}\right),$$

that of $T_n(x)$, which begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & -3 & 0 & 4 & 0 & 0 & \dots \\ 1 & 0 & -8 & 0 & 8 & 0 & \dots \\ 0 & 5 & 0 & -20 & 0 & 16 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \quad \text{(A053120)}$$

is not a Riordan array. However the Riordan array

$$\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)$$

which begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ -2 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & -3 & 0 & 1 & 0 & 0 & \dots \\ 2 & 0 & -4 & 0 & 1 & 0 & \dots \\ 0 & 5 & 0 & -5 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \quad \text{(A108045)}$$

is the coefficient array for the orthogonal polynomials given by $(2-0^n)T_n(x/2)$.

Orthogonal polynomials can also be defined in terms of the three term recurrence that they obey. Thus, for instance,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),$$

with a similar recurrence for $U_n(x)$. Of course, we then have

$$U_n(x/2) = xU_{n-1}(x/2) - U_{n-2}(x/2),$$

for instance. This last recurrence corresponds to the fact that the production matrix of $\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} = (c(x^2), xc(x^2))$ is given by

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Note that many of the above results can also be found in [18]

6 The Boubaker polynomials

The Boubaker polynomials arose from the discretization of the equations of heat transfer in pyrolysis [2, 14, 16] starting from an assumed solution of the form

$$\frac{1}{N}e^{\frac{A}{\frac{H}{z}+1}}\sum_{m=0}^{\infty}\xi_m J_m(t)$$

where J_m is the m-th order Bessel function of the first kind. Upon truncation, we get a set of equations

$$\begin{aligned}
Q_1(z)\xi_0 &= \xi_1 \\
Q_1(z)\xi_1 &= -2\xi_0+\xi_2 \\
\dots & \\
Q_1(z)\xi_m &= \xi_{m-1}+\xi_{m+1} \\
\dots &
\end{aligned}$$

with coefficient matrix

$$\left(\begin{array}{ccccccc}
0 & 1 & 0 & 0 & 0 & 0 & \dots \\
-2 & 0 & 1 & 0 & 0 & 0 & \dots \\
0 & 1 & 0 & 1 & 0 & 0 & \dots \\
0 & 0 & 1 & 0 & 1 & 0 & \dots \\
0 & 0 & 0 & 1 & 0 & 1 & \dots \\
0 & 0 & 0 & 0 & 1 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

in which we recognize the production matrix of the Riordan array

$$\left(\frac{1+3x^2}{1+x^2},\frac{x}{1+x^2}\right)^{-1}.$$

The *Boubaker polynomials* $B_n(x)$ are defined to be the family of orthogonal polynomials with coefficient array given by

$$\left(\frac{1+3x^2}{1+x^2},\frac{x}{1+x^2}\right).$$

We have $B_0(x)=1$ and

$$B_n(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}\frac{n-4k}{n-k}(-1)^k x^{n-2k}, \quad n>0. \tag{8}$$

We also have

$$\sum_{n=0}^{\infty}B_n(x)t^n=\frac{1+3t^2}{1-xt+t^2}.$$

The connection to Riordan arrays has already been noted in [19]. The matrix $\left(\frac{1+3x^2}{1+x^2}, \frac{x}{1+x^2}\right)$ begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ -2 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & -3 & 0 & -2 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

and hence we have

$$\begin{aligned} B_0(x) &= 1 \\ B_1(x) &= x \\ B_2(x) &= x^2+2 \\ B_3(x) &= x^3+1 \\ B_4(x) &= x^4-2 \\ B_5(x) &= x^5-x^3-3x, \dots \end{aligned}$$

We can find an expression for the general term of the Boubaker coefficient matrix $\left(\frac{1+3x^2}{1+x^2}, \frac{x}{1+x^2}\right)$ as follows. We have

$$\begin{aligned} \left(\frac{1+3x^2}{1+x^2}, \frac{x}{1+x^2}\right) &= (1+3x^2, x) \cdot \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right) \\ &= \left(3\binom{2}{n-k} - 6\binom{1}{n-k} + 4\binom{0}{n-k}\right) \cdot \left((-1)^{\frac{n-k}{2}} \binom{\frac{n+k}{2}}{k} \frac{1+(-1)^{n-k}}{2}\right), \end{aligned}$$

where $3\binom{2}{n} - 6\binom{1}{n} + 4\binom{0}{n}$ represents the general term of the sequence $1, 0, 3, 0, 0, 0, \dots$ with g.f. $1+3x^2$. Thus the general term of the Boubaker coefficient array is given by

$$\sum_{j=0}^{n} \left(3\binom{2}{n-j} - 6\binom{1}{n-j} + 4\binom{0}{n-j}\right) \left((-1)^{\frac{j-k}{2}} \binom{\frac{j+k}{2}}{k} \frac{1+(-1)^{j-k}}{2}\right).$$

7 The family of Chebyshev-Boubaker polynomials

Inspired by the foregoing, we now define the Chebyshev-Boubaker polynomials with parameters (r, s) to be the orthogonal polynomials $B_n(x; r, s)$ whose coefficient array is the Riordan array

$$\mathfrak{B} = \left(\frac{1+rx+sx^2}{1+x^2}, \frac{x}{1+x^2}\right).$$

That these are orthogonal polynomials is a consequence of the fact that the production array of $\mathfrak{B}^{-1}$ is the tridiagonal matrix

$$\left(\begin{array}{ccccccc} -r & 1 & 0 & 0 & 0 & 0 & \dots \\ 1-s & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We immediately note the factorization

$$\mathfrak{B} = (1+rx+sx^2, x) \cdot \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right). \tag{9}$$

It is clear that we have

$$\sum_{n=0}^{\infty} B_n(x; r, s)t^n = \frac{1+rt+st^2}{1-xt+t^2}.$$

We have

$$\begin{aligned} B_n(x; 0, 0) &= U_n(x/2), \\ B_n(x; 0, -1) &= (2-0^n) \cdot T_n(x/2), \\ B_n(x; 0, 3) &= B_n(x). \end{aligned}$$

We can characterize $B_n(x; r, s)$ in terms of the Chebyshev polynomials as follows.

Proposition 23.

$$B_n(x; r, s) = U_n(x/2) + rU_{n-1}(x/2) + sU_{n-2}(x/2). \tag{10}$$

Proof. This follows from the definition since $\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)$ is the coefficient array for $U_n(x/2)$. □

Proposition 24.

$$B_n(x; r, s) = rU_{n-1}(x/2) + (s+1)U_{n-2}(x/2) + 2T_n(x/2) - 0^n. \tag{11}$$

Proof. This follows since

$$\frac{1+rx+sx^2}{1+x^2} = r\frac{x}{1+x^2} + (s+1)\frac{x^2}{1+x^2} + \frac{1-x^2}{1+x^2}.$$

□

Proposition 25.

$$B_n(x; r, s) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n-(s+1)k}{n-k} (-1)^k x^{n-2k} + rU_{n-1}(x/2). \tag{12}$$

Proof. Indeed, the polynomials defined by

$$\left(\frac{1+sx^2}{1+x^2}, \frac{x}{1+x^2}\right)$$

are given by

$$B_n(x;0,s) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n-(s+1)k}{n-k} (-1)^k x^{n-2k}.$$

This can be shown in a similar fashion to Eq. (8). □

We can also use the factorization in Eq. (9) to derive another expression for these polynomials. The general term of the Riordan array $\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)$ is given by

$$a_{n,k} = (-1)^{\frac{n-k}{2}} \binom{\frac{n+k}{2}}{k} \frac{1+(-1)^{n-k}}{2},$$

while the general term of the array $(1+rx+sx^2, x)$ is given by $f(n-k)$, where $f(n)$ can be expressed, for instance, as

$$f(n) = (s-r+1)\binom{0}{n} + (r-2s)\binom{1}{n} + s\binom{2}{n}.$$

Thus the general element of $\mathfrak{B}$ is given by

$$\sum_{j=0} f(n-j) a_{j,k} = \sum_{j=0} ((s-r+1)\binom{0}{n-j} + (r-2s)\binom{1}{n-j} + s\binom{2}{n-j})(-1)^{\frac{j-k}{2}} \binom{\frac{j+k}{2}}{k} \frac{1+(-1)^{j-k}}{2}.$$

We finish this section by noting that we could have defined a more general family of Chebyshev-Boukaber orthogonal polynomials as follows: Let

$$\mathcal{B}_{r,s,\alpha,\beta} = \left(\frac{1+rx+sx^2}{1+\alpha x+\beta x^2}, \frac{x}{1+\alpha x+\beta x^2}\right).$$

Then this array is the coefficient array for the polynomials $B(n;r,s;\alpha,\beta)$. This is a family of orthogonal polynomials since the production array of $\mathcal{B}^{-1}_{r,s,\alpha,\beta}$ is given by

$$\begin{pmatrix} \alpha-r & 1 & 0 & 0 & 0 & 0 & \dots \\ \beta-s & \alpha & 1 & 0 & 0 & 0 & \dots \\ 0 & \beta & \alpha & 1 & 0 & 0 & \dots \\ 0 & 0 & \beta & \alpha & 1 & 0 & \dots \\ 0 & 0 & 0 & \beta & \alpha & 1 & \dots \\ 0 & 0 & 0 & 0 & \beta & \alpha & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have

Proposition 26.

$$B(n;r,s;\alpha,\beta) = (\sqrt{\beta})^n U_n\left(\frac{x-\alpha}{2\sqrt{\beta}}\right) + r(\sqrt{\beta})^{n-1} U_{n-1}\left(\frac{x-\alpha}{2\sqrt{\beta}}\right) + s(\sqrt{\beta})^{n-2} U_{n-2}\left(\frac{x-\alpha}{2\sqrt{\beta}}\right).$$

8 The inverse matrix $\mathfrak{B}^{-1}$

We recall that the first column of $\mathfrak{B}^{-1}$ contains the moment sequence for the weight function defined by the Chebyshev-Boubaker polynomials $B(n;r,s)$. In this section, we shall be interested in studying this sequence, including its Hankel transform, as well as looking at the row sums, and (more briefly) the diagonal sums, of $\mathfrak{B}^{-1}$.

The inverse of the matrix $\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)$, corresponding to $r = s = 0$, is the much-studied Catalan related matrix

$$(c(x^2), xc(x^2)), \quad c(x) = \frac{1-\sqrt{1-4x}}{2x},$$

where $c(x)$ is the generating function of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ A000108. The inverse of $\left(\frac{1-x^2}{1+x^2}, \frac{x}{1+x^2}\right)$, which corresponds to $r = 0$, $s = -1$, is the matrix

$$\left(\frac{1}{\sqrt{1-4x^2}}, xc(x^2)\right),$$

which again finds many applications. It is A108044.

Using the theory of Riordan arrays, we find that

$$\begin{aligned}
\mathfrak{B}^{-1} &= \left(\frac{1+rx+sx^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \\
&= \left(\frac{1+x^2c(x^2)^2}{1+rxc(x^2)+sx^2c(x^2)^2}, xc(x^2)\right) \\
&= \left(\frac{c(x^2)}{1+rxc(x^2)+sx^2c(x^2)^2}, xc(x^2)\right).
\end{aligned}$$

Note also that

$$\begin{aligned}
\mathfrak{B}^{-1} &= \left(\frac{1+rx+sx^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \\
&= \left((1+rx+sx^2, x)\cdot\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)\right)^{-1} \\
&= \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}\cdot(1+rx+sx^2, x)^{-1} \\
&= (c(x^2), xc(x^2))\cdot\left(\frac{1}{1+rx+sx^2}, x\right).
\end{aligned}$$

Thus for instance the generating function of the first column of the inverse matrix is

$$\frac{c(x^2)}{1+rxc(x^2)+sx^2c(x^2)^2} = \frac{1-\sqrt{1-4x^2}}{s+rx+2(1-s)x^2+(s+rx)\sqrt{1-4x^2}} \tag{13}$$

$$= \frac{1+s+2rx+(s-1)\sqrt{1-4x^2}}{2(s+r(s+1)x+(r^2+(s-1)^2)x^2}. \tag{14}$$

We can find expressions for the general term u_n of the sequence given by the first column of $\mathfrak{B}^{-1}$ and the general term v_n of the row sums $\mathfrak{B}^{-1}$ as follows. We start with

$$\mathfrak{B}^{-1} = (c(x^2), xc(x^2)) \cdot \left(\frac{1}{1+rx+sx^2}, x\right).$$

The first matrix has general element

$$\binom{n+1}{\frac{n-k}{2}} \frac{k+1}{n+1} \frac{1+(-1)^{n-k}}{2},$$

while the second matrix is the sequence (Appell) array for the generalized Fibonacci numbers

$$F_{r,s}(n) = [x^n]\frac{1}{1+rx+sx^2} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (-r)^{n-2i}(-s)^k.$$

Thus the general term of $\mathfrak{B}^{-1}$ is given by

$$T_{n,k} = \sum_{j=0}^{n} \binom{n+1}{\frac{n-j}{2}} \frac{j+1}{n+1} \frac{1+(-1)^{n-j}}{2} \sum_{i=0}^{\lfloor \frac{j-k}{2} \rfloor} \binom{j-k-i}{i} (-r)^{j-k-2i}(-s)^i. \tag{15}$$

Setting $k=0$ we obtain

$$u_n = \sum_{j=0}^{n} \binom{n+1}{\frac{n-j}{2}} \frac{j+1}{n+1} \frac{1+(-1)^{n-j}}{2} \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j-i}{i} (-r)^{j-2i}(-s)^i. \tag{16}$$

This last equation translates the fact that

$$u_n = [x^n]\frac{c(x^2)}{1+rxc(x^2)+sx^2c(x^2)^2} = [x^n](c(x^2), xc(x^2)) \cdot \frac{1}{1+rx+sx^2}.$$

Note that another expression for u_n is given by

$$u_n = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \left\{ \binom{n}{k} - \binom{n}{k-1} \right\} \sum_{j=0}^{\lfloor \frac{n-2k}{2} \rfloor} \binom{n-2k-j}{j} (-r)^{n-2k-2j}(-b)^j,$$

which represents u_n as the diagonal sums of the Hadamard product of the reversal of $(c(x), xc(x))$ and the sequence array of $F_{r,s}(n)$.

The general term v_n of the row sums of $\mathfrak{B}^{-1}$ is given by

$$v_n = \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n+1}{\frac{n-j}{2}} \frac{j+1}{n+1} \frac{1+(-1)^{n-j}}{2} \sum_{i=0}^{\lfloor \frac{j-k}{2} \rfloor} \binom{j-k-i}{i} (-r)^{j-k-2i}(-s)^i.$$

We can deduce from Eq. (13) that the weight function for the first column $\{u_n\}$ of the inverse is given by

$$-\frac{1}{2\pi}\frac{(s-1)\sqrt{4-x^2}}{(1-s)^2+r^2+r(s+1)x+sx^2}+\alpha(r,s)\delta_{\left(\frac{(1-s)\sqrt{r^2-4s}}{2s}-\frac{r(s+1)}{2s}\right)}$$
$$+\beta(r,s)\delta_{\left(\frac{(s-1)\sqrt{r^2-4s}}{2s}-\frac{r(s+1)}{2s}\right)},$$

for appropriate values of $\alpha(r,s)$ and $\beta(r,s)$. Thus for $r=1$, $s=2$, we find that the terms of the first column have integral representation

$$u_n=-\frac{1}{2\pi}\int_{-2}^{2}\frac{x^n\sqrt{4-x^2}}{2+3x+2x^2}\,dx+\left(\frac{3}{4}-\frac{1}{4\sqrt{7}}i\right)\left(-\frac{3}{4}-\frac{\sqrt{7}}{4}i\right)^n+\left(\frac{3}{4}+\frac{1}{4\sqrt{7}}i\right)\left(-\frac{3}{4}+\frac{\sqrt{7}}{4}i\right)^n,$$

while for $r=2$, $s=3$, we find that the terms of the first column have integral representation

$$u_n=-\frac{1}{\pi}\int_{-2}^{2}\frac{x^n\sqrt{4-x^2}}{8+8x+3x^2}\,dx+\left(\frac{2}{3}-\frac{\sqrt{2}}{6}i\right)\left(-\frac{4}{3}-\frac{2\sqrt{2}}{3}i\right)^n+\left(\frac{2}{3}+\frac{\sqrt{2}}{6}i\right)\left(-\frac{4}{3}+\frac{2\sqrt{2}}{3}i\right)^n.$$

Using the techniques of [1, 6] we can prove the following.

Proposition 27. *The Hankel transform of the first column of*

$$\left(\frac{1+rx+sx^2}{1+x^2},\frac{x}{1+x^2}\right)^{-1}$$

is given by $h_n=(1-s)^n$.

We can recouch this result in the following terms:

Proposition 28. *The moments of the Chebyshev-Boubaker polynomials* $B_n(x;r,s)$ *have Hankel transform equal to* $(1-s)^n$.

Recall that the elements of the first column of $\mathfrak{B}^{-1}$ are the moments for the density measure associated to the polynomials $B_n(x;r,s)$. We also have

Proposition 29. *The Hankel transform of the row sums of the Riordan matrix*

$$\left(\frac{1+rx+sx^2}{1+x^2},\frac{x}{1+x^2}\right)^{-1}$$

is given by

$$h_n=[x^n]\frac{1}{1+(s-r-1)x+s^2x^2}.$$

Proof. The g.f. of the row sums is given by

$$\frac{\frac{c(x^2)}{1+rxc(x^2)+sx^2c(x^2)^2}}{1-xc(x^2)}=\frac{1+s+2rx+(s-1)\sqrt{1-4x^2}}{2(s+r(s+1)x+(r^2+(s-1)^2)x^2}\frac{1-2x+\sqrt{1-4x^2}}{2(1-2x)}.$$

The result again follows from the techniques of [1, 6]. □

We note that the g.f. of the row sums may be written as

$$\frac{1}{1+rxc(x^2)+sx^2c(x^2)^2}\frac{c(x^2)}{1-xc(x^2)} = \frac{c(x^2)}{1+rxc(x^2)+sx^2c(x^2)^2}\frac{1}{1-xc(x^2)}.$$

Thus the row sums of the inverse matrix are a convolution of

$$[x^n]\frac{1}{1+rxc(x^2)+sx^2c(x^2)^2}$$

and the central binomial coefficients $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ A001405, or alternatively a convolution of

$$[x^n]\frac{c(x^2)}{1+rxc(x^2)+sx^2c(x^2)^2}$$

and

$$\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + 0^n.$$

Example 30. The Hankel transforms of the row sums of the inverse matrices

$$\left(\frac{1+x-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1} \quad \text{and} \quad \left(\frac{1+3x+x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}$$

are both given by F_{2n+2}.

In the case of the matrix

$$\left(\frac{1+x-x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}$$

the row sums are expressible as

$$\left([x^n]\frac{\sqrt{1-4x^2}-x}{1-5x^2}\right)*\left(\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}+0^n\right) = ((-1)^n \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor}\left(\binom{n}{k}-\binom{n}{k-1}\right)F_{n-2k+1})*\left(\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}+0^n\right),$$

where the first element of the convolution is $(-1)^n$ A098615(n). Note that these constituent sequences have Hankel transforms of 2^n and $1, 0, -1, 0, 1, 0, -1, 0, \ldots$, respectively. Alternatively the row sums are given by

$$\left([x^n]\frac{1-x-4x^2+(1-x)\sqrt{1-4x^2}}{2(1-5x^2)}\right)*\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

In this case, the Hankel transforms of the constituent elements of the convolution are given by $1, 1, 1, 1, 0, 0, -1, -1, -1-1, 0, \ldots$, and the all 1's sequence. For the case of the matrix

$$\left(\frac{1+3x+x^2}{1+x^2}, \frac{x}{1+x^2}\right)^{-1},$$

we note that

$$\frac{c(x^2)}{1+3xc(x^2)+x^2c(x^2)^2} = \frac{1}{1+3x}$$

and so the row sums of the inverse in this case are simply given by

$$\sum_{k=0}^{n}(-3)^{n-k}\left(\binom{k-1}{\lfloor\frac{k-1}{2}\rfloor}+0^k\right).$$

We finish this section by noting that the diagonal sums of $\mathfrak{B}^{-1}$ are also of interest. They have generating function

$$\frac{1+rx-2(1-s)x^2-(1+rx)\sqrt{1-4x^2}}{2x^2(s+rx(1+s)+(r^2+(s-1)^2)x^2)}.$$

For instance, the diagonal sums of

$$\left(\frac{1-x-x^2}{1+x^2},\frac{x}{1+x^2}\right)^{-1},$$

which begin

$$1,1,4,6,18,32,85,165,411,839,2013,\ldots,$$

have as Hankel transform the 12-period sequence with g.f. $\frac{1+3x+x^2-x^3}{1-x^2+x^4}$ which begins

$$1,3,2,2,1,-1,-1,-3,-2,-2,-1,1,1,3,2,2,1,-1,-1,-3,-2,\ldots$$

Similarly, the diagonal sums of the matrix

$$\left(\frac{1-2x-3x^2}{1+x^2},\frac{x}{1+x^2}\right)^{-1},$$

which begin

$$1,2,9,26,94,300,1025,3370,11322,\ldots,$$

have as Hankel transform the sequence with g.f. $\frac{1+5x+3x^2-x^3}{(1+x^2)^2}$ and general term

$$(1-n)\cos\left(\frac{\pi n}{2}\right)+(3n+2)\sin\left(\frac{\pi n}{2}\right).$$

We can conjecture that the Hankel transform of the diagonal sums of $\mathfrak{B}^{-1}$ in the general case is given by

$$[x^n]\frac{1+(2-s)x-sx^2-x^3}{1+(r^2-2)x^2+x^4}.$$

9 A curious relation

The third column of the Boubaker coefficient matrix $\left(\frac{1+3x^2}{1+x^2},\frac{x}{1+x^2}\right)$ has general term

$$t_n=\sum_{j=0}^{n}\left(3\binom{2}{n-j}-6\binom{1}{n-j}+4\binom{0}{n-j}\right)(-1)^{\frac{j-2}{2}}\binom{\frac{j+2}{2}}{2}\frac{1+(-1)^{j-2}}{2}.$$

This sequence begins

$$0, 0, 1, 0, 0, 0, -3, 0, 8, 0, -15, 0, 24, 0, -35, 0, 48, 0, -63, 0, 80, \ldots$$

Now the sequence t_{2n+2} is therefore given by

$$1, 0, -3, 8, -15, 24, -35, 48, \ldots$$

This is A131386, with general term $(1-n^2)(-1)^n$. The interested reader may wish to verify that

$$t_{2n+2} = \frac{1}{2\pi}\Re \int_{-2}^{2} \left(\frac{1+x}{1-x}\right)^n \sqrt{4-x^2}\, dx$$

(here, $\Re$ returns the real part of a complex number).

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