A Note on Three Families of Orthogonal Polynomials defined by Circular Functions, and Their Moment Sequences

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Abstract

Using the language of exponential Riordan arrays, we study three distinct families of orthogonal polynomials defined by trigonometric functions. We study the moment sequences of theses families, finding continued fraction expressions for their generating functions, and calculate the Hankel transforms of these moment sequences. Results related to the Euler or zigzag numbers, as well as the generalized Euler or Springer numbers, are found. In addition, we characterize the Dowling numbers as moments of a family of orthogonal polynomials.

Riordan arrays [12] are known to have many applications in the area of combinatorics. They often allow us to express in concise form results about objects of combinatorial interest [14], and to infer analogues and generalizations, by virtue of their expressive form. In this note, we continue a study of the links between certain Riordan arrays and orthogonal polynomials [3, 7], and study three families of orthogonal polynomials each defined by trigonometric functions. Our results stem from the observation that if a Riordan array L has a tri-diagonal production matrix, then L^{-1} is the coefficient array of a family of orthogonal polynomials [1, 2]. We shall be concerned in this note with exponential Riordan arrays [4], that is, infinite lower-triangular matrices M (denoted by $[g, f]$) defined by a pair of power series $f(x)$, where $f(0) = 0$, $f'(0) \neq 0$, and $g(x)$, where $g(0) \neq 0$, such that the k-th column of M has exponential generating function $g(x)(f(x))^k/k!$. We recall that for an exponential Riordan array

$$L = [g, f]$$

the production matrix P_L of L [4, 5, 6] is the matrix

$$P_L = L^{-1}\tilde{L},$$

where $\tilde{L}$ is the matrix L with the first row removed. The bivariate generating function of P_L is given by

$$e^{xy}(c(x) + yr(x)),$$

where we have the Deutsch equations

$$r(x) = f'(\bar{f}(x)), \tag{1}$$

and

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))}. \tag{2}$$

The following well-known results (the first is the well-known "Favard's Theorem"), which we essentially reproduce from [8], specify the links between orthogonal polynomials, the three-term recurrences that define them, the recurrence coefficients of those three-term recurrences, and the g.f. of the moment sequence of the orthogonal polynomials.

Theorem 1. *[8] (Cf. [16, Théorème 9 on p. I-4], or [17, Theorem 50.1]). Let $(p_n(x))_{n\ge 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree $n = 0, 1, \ldots$ Then the sequence $(p_n(x))$ is (formally) orthogonal if and only if there exist sequences $(\alpha_n)_{n\ge 0}$ and $(\beta_n)_{n\ge 1}$ with $\beta_n \neq 0$ for all $n \geq 1$, such that the three-term recurrence*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \text{for} \quad n \geq 1,$$

holds, with initial conditions $p_0(x) = 1$ and $p_1(x) = x - \alpha_0$.

Theorem 2. *[8] (Cf. [16, Proposition 1, (7), on p. V-5], or [17, Theorem 51.1]). Let $(p_n(x))_{n\ge 0}$ be a sequence of monic polynomials, which is orthogonal with respect to some functional $\mathcal{L}$. Let*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \text{for} \quad n \geq 1,$$

be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the generating function

$$g(x) = \sum_{k=0}^{\infty} \mu_k x^k$$

for the moments $\mu_k = \mathcal{L}(x^k)$ satisfies

$$g(x) = \cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}.$$

The *Hankel transform* [10] of a given sequence $A = \{a_0, a_1, a_2, ...\}$ is the sequence of Hankel determinants $\{h_0, h_1, h_2, \ldots\}$ where $h_n = |a_{i+j}|_{i,j=0}^{n}$, i.e.,

$$A = \{a_n\}_{n\in\mathbb{N}_0} \quad \rightarrow \quad h = \{h_n\}_{n\in\mathbb{N}_0}: \quad h_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{vmatrix}. \tag{3}$$

The Hankel transform of a sequence a_n and that of its binomial transform are equal. In the case that a_n has g.f. $g(x)$ expressible in the form

$$g(x) = \cfrac{a_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}$$

then we have [8]

$$h_n = a_0^{n+1} \beta_1^n \beta_2^{n-1} \cdots \beta_{n-1}^2 \beta_n = a_0^{n+1} \prod_{k=1}^n \beta_k^{n+1-k}. \tag{4}$$

Note that this is independent of α_n.
The exponential Riordan arrays that we shall study in the sequel will have polynomial entries, where the polynomials have integer coefficients.

1 The exponential Riordan array $L = \left[\frac{1}{\cos^r x}, \frac{\sin x}{\cos x}\right]$

The numbers with generating function $\frac{1}{\cos x} = \sec x$ begin

$$1, 0, 1, 0, 5, 0, 61, 0, \ldots$$

The "unaerated" sequence $1, 1, 5, 61, \ldots$ is called the sequence of Euler, secant, or Zig numbers A000364. Both sequences are of importance in combinatorics. For instance, they are closely associated to alternating permutations.
We consider the related exponential Riordan array

$$L = \left[\frac{1}{\cos^r x}, \frac{\sin x}{\cos x}\right] = [\sec^r x, \tan x],$$

which depends on the parameter r. The inverse of L is given by

$$L^{-1} = \left[\frac{1}{(1+x^2)^{\frac{r}{2}}}, \tan^{-1} x\right].$$

We have the following proposition, relating these matrices to a family of orthogonal polynomials.

Proposition 3. *The matrix*

$$L^{-1} = \left[\frac{1}{(1+x^2)^{\frac{r}{2}}}, \tan^{-1} x\right]$$

is the coefficient array of the family of orthogonal polynomials $P_n^{(r)}(x)$ which satisfy the following three-term recurrence

$$P_n^{(r)}(x) = x P_{n-1}^{(r)}(x) - ((n-1)r + (n-1)(n-2)) P_{n-2}^{(r)}(x),$$

with $P_0^{(r)}(x) = 1$, $P_1^{(r)}(x) = x$.

Proof. We must show that the production matrix of L is tri-diagonal. We have $f(x) = \tan x$ and thus $\bar{f}(x) = \tan^{-1}(x)$, and $f'(x) = \sec^2 x$. Similarly $g(x) = \sec^r x$ and hence $g'(x) = r \sin x \sec^{r+1} x$. Thus

$$r(x) = f'(\bar{f}(x)) = 1 + x^2,$$

while

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = rx.$$

This implies that the production matrix P_L of L is generated by

$$e^{xy}(rx + (1 + x^2)y).$$

Thus the production matrix P_L has the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ r & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2r+2 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3r+6 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4r+12 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 5r+20 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which proves the assertion. □

Note that we can write the above three-term recurrence as

$$P_n^{(r)}(x) = xP_{n-1}^{(r)}(x) - (n-1)(n+r-2)P_{n-2}^{(r)}(x).$$

Corollary 4. *The generating function of the moment sequence associated to the family of orthogonal polynomials $P_n^{(r)}$ is given by*

$$\sec^r x = \cfrac{1}{1 - \cfrac{rx^2}{1 - \cfrac{2(r+1)x^2}{1 - \cfrac{3(r+2)x^2}{1 - \cfrac{4(r+3)x^2}{1 - \cdots}}}}}.$$

We note that this moment sequence begins

$$1, 0, r, 0, r(3r+2), 0, r(15r^2 + 30r + 16), 0, \ldots.$$

This sequence appears as the first column of the matrix L.

Corollary 5. *The Hankel transform of the moment sequence associated to the family of orthogonal polynomials $P_n^{(r)}$ is given by*

$$h_n = \prod_{k=0}^{n} k!(r+k)^{n-k}.$$

Proof. We have

$$h_n = \prod_{k=0}((k+1)r + k(k+1))^{n-k} = \prod_{k=0}^{n}(k+1)^{n-k}(r+k)^{n-k} = \prod_{k=0}^{n} k!(r+k)^{n-k}.$$

□

Setting $r = 1$ shows that $\sec x$ is the generating function of the moments of the orthogonal polynomials

$$P_n^{(1)}(x) = xP_{n-1}^{(1)}(x) - (n-1)^2 P_{n-2}^{(1)}(x).$$

The Hankel transform of these moments (the aerated Euler numbers) is thus given by

$$h_n = \prod_{k=0}^{n} k!(k+1)^{n-k} = \prod_{k=0}^{n} k!^2.$$

This is A055209.

2 The exponential Riordan array $L = \left[\frac{1}{(\cos x - \sin x)^r}, \frac{\sin x}{\cos x - \sin x}\right]$

We now modify the denominator in the foregoing from $\cos x$ to $\cos x - \sin x$. We note that

$$\frac{1}{\cos x - \sin x}$$

is the generating function of the so-called Springer, or generalized Euler numbers A001586. Thus we are led to consider the exponential Riordan array

$$L = \left[\frac{1}{(\cos x - \sin x)^r}, \frac{\sin x}{\cos x - \sin x}\right] = \left[\frac{1}{(\cos x - \sin x)^r}, \frac{1}{1 - \tan x} - 1\right].$$

Note that

$$L = [(f'(x))^{r/2}, f(x)] \quad \text{where} \quad f(x) = \frac{\sin x}{\cos x - \sin x}.$$

Again, we find that

$$L^{-1} = \left[\frac{1}{(1 + 2x + 2x^2)^{\frac{r}{2}}}, \tan^{-1}\left(\frac{x}{1+x}\right)\right]$$

is the coefficient array of a family of orthogonal polynomials. This is the content of

Proposition 6. *The matrix*

$$L^{-1} = \left[\frac{1}{(1 + 2x + 2x^2)^{\frac{r}{2}}}, \tan^{-1}\left(\frac{x}{1+x}\right)\right]$$

is the coefficient array of the family of orthogonal polynomials $P_n^{(r)}(x)$ which satisfy the following three-term recurrence

$$P_n^{(r)}(x) = (x - (r + 2(n-1)))P_{n-1}^{(r)}(x) - (2(n-1)r + 2(n-1)(n-2))P_{n-2}^{(r)}(x),$$

with $P_0^{(r)}(x) = 1$, $P_1^{(r)}(x) = x - r$.

Proof. We have $g(x) = \frac{1}{(\cos x - \sin x)^r}$, and $f(x) = \frac{\sin x}{\cos x - \sin x}$. Then using equations (1) and (2) we obtain that the generating function of the production array P_L of L is equal to

$$e^{xy}(r(1+2x) + (1+2x+2x^2)y).$$

This implies that P_L has the form

$$\begin{pmatrix} r & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2r & r+2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4r+4 & r+4 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 6r+12 & r+6 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 8r+24 & r+8 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 10r+40 & r+10 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which proves the assertion. □

Corollary 7. *The generating function of the moment sequence associated to the family of orthogonal polynomials $P_n^{(r)}$ is given by*

$$\frac{1}{(\cos x - \sin x)^r} = \cfrac{1}{1 - rx - \cfrac{2rx^2}{1-(r+2)x - \cfrac{4(r+1)x^2}{1-(r+4)x - \cfrac{6(r+2)x^2}{1-(r+6)x - \cfrac{8(r+3)x^2}{1-\cdots}}}}}.$$

We note that this moment sequence begins

$$1, r, r^2+2r, r(r^2+6r+4), r(r^3+12r^2+28r+16), \ldots.$$

Corollary 8. *The Hankel transform of the moment sequence associated to the family of orthogonal polynomials $P_n^{(r)}$ is given by*

$$h_n = 2^{\binom{n+1}{2}} \prod_{k=0}^{n} k!(r+k)^{n-k}.$$

Proof. We have

$$h_n = \prod_{k=0}^{n} 2^k k!(r+k)^{n-k} = 2^{\binom{n+1}{2}} \prod_{k=0}^{n} k!(r+k)^{n-k}.$$

□

Setting $r=1$ shows that $\frac{1}{\cos x - \sin x}$ is the generating function of the moments of the orthogonal polynomials

$$P_n^{(1)}(x) = (x-(2n-1))P_{n-1}^{(1)}(x) - 2(n-1)^2 P_{n-2}^{(1)}(x).$$

The Hankel transform of these moments (the generalized Euler or Springer numbers) is thus given by

$$h_n = 2^{\binom{n+1}{2}} \prod_{k=0}^{n} k!(k+1)^{n-k} = 2^{\binom{n+1}{2}} \prod_{k=0}^{n} k!^2.$$

This is A091804.
It is of interest to analyze the structure of the moment sequence

$$1, r, r^2+2r, r(r^2+6r+4), r(r^3+12r^2+28r+16), \ldots.$$

This is a sequence of polynomials in r, with coefficient array given by the exponential Riordan array

$$\left[1, \ln\left(\frac{1}{\cos x - \sin x}\right)\right].$$

We can generalize the foregoing results as follows.

Proposition 9. *The matrix*

$$L^{-1} = \left[\frac{1}{(1+2x+2sx^2)^{\frac{r}{2}}}, \frac{\tan^{-1}\left(\frac{\sqrt{2s-1}x}{1+x}\right)}{\sqrt{2s-1}}\right]$$

is the coefficient array of the family of orthogonal polynomials $P_n^{(r,s)}(x)$ which satisfy the following three-term recurrence

$$P_n^{(r,s)}(x) = (x-(r+2(n-1)))P_{n-1}^{(r,s)}(x) - (2(n-1)r+2s(n-1)(n+r-2))P_{n-2}^{(r,s)}(x),$$

with $P_0^{(r,s)}(x) = 1$, $P_1^{(r,s)}(x) = x - r$.

In fact, we find that the matrix L, where

$$L = \left[\frac{1}{(\cos(\sqrt{2s-1}x) - \sin(\sqrt{2s-1}x)/\sqrt{2s-1})^r}, \frac{\sin(\sqrt{2s-1}x)/\sqrt{2s-1}}{\cos(\sqrt{2s-1}x) - \sin(\sqrt{2s-1}x)/\sqrt{2s-1}}\right],$$

has production matrix

$$\begin{pmatrix} r & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2sr & r+2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4s(r+1) & r+4 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 6s(r+2) & r+6 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 8s(r+3) & r+8 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 10s(r+4) & r+10 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which shows that L^{-1} is indeed the coefficient array of the family of orthogonal polynomials $P_n^{(r,s)}$. In addition, we see that the generating function

$$\frac{1}{(\cos(\sqrt{2s-1}x) - \sin(\sqrt{2s-1}x)/\sqrt{2s-1})^r}$$

has the following continued fraction expression:

$$\cfrac{1}{1-rx-\cfrac{2srx^2}{1-(r+2)x-\cfrac{4s(r+1)x^2}{1-(r+4)x-\cfrac{6s(r+2)x^2}{1-(r+6)x-\cfrac{8s(r+3)x^2}{1-\cdots}}}}}.$$

Hence the Hankel transform of the moment sequence of the family of orthogonal polynomials $P_n^{(r,s)}(x)$ is given by

$$h_n(r,s) = (2s)^{\binom{n+1}{2}} \prod_{k=0}^{n} k!(r+k)^{n-k}.$$

Writing $u = 2s-1$, we have

Proposition 10. *The matrix*

$$L^{-1} = \left[\frac{1}{(1+2x+(u+1)x^2)^{\frac{r}{2}}}, \frac{\tan^{-1}\left(\frac{\sqrt{u}x}{1+x}\right)}{\sqrt{u}}\right]$$

is the coefficient array of the family of orthogonal polynomials $P_n^{(r,u)}(x)$ which satisfy the following three-term recurrence

$$P_n^{(r,s)}(x) = (x-(r+2(n-1)))P_{n-1}^{(r,s)}(x) - (n-1)(n+r-2)(u+1)P_{n-2}^{(r,s)}(x),$$

with $P_0^{(r,s)}(x) = 1$, $P_1^{(r,s)}(x) = x-r$.

In this case, the production matrix of L is given by

$$\left(\begin{array}{ccccccc}
r & 1 & 0 & 0 & 0 & 0 & \cdots \\
r(u+1) & r+2 & 1 & 0 & 0 & 0 & \cdots \\
0 & 2(r+1)(u+1) & r+4 & 1 & 0 & 0 & \cdots \\
0 & 0 & 3(r+2)(u+1) & r+6 & 1 & 0 & \cdots \\
0 & 0 & 0 & 4(r+3)(u+1) & r+8 & 1 & \cdots \\
0 & 0 & 0 & 0 & 5(r+4)(u+1) & r+10 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

We note that for orthogonality, we must have $s \neq 0$, or equivalently, $u \neq -1$. The case $r=1$, $s=0$ is interesting. In this case, we have

$$L^{-1} = \left[\frac{1}{\sqrt{1+2x}}, \frac{\ln(1+2x)}{2}\right] = \left[\frac{1}{\sqrt{1+2x}}, \ln\left(\sqrt{1+2x}\right)\right].$$

The matrix L^{-1}, which in this case is not the coefficient array of a family of orthogonal polynomials, begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & -4 & 1 & 0 & 0 & 0 & \cdots \\ -15 & 23 & -9 & 1 & 0 & 0 & \cdots \\ 105 & -176 & 86 & -16 & 1 & 0 & \cdots \\ -945 & 1689 & -950 & 230 & -25 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

with L given by

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 13 & 9 & 1 & 0 & 0 & \cdots \\ 1 & 40 & 58 & 16 & 1 & 0 & \cdots \\ 1 & 121 & 330 & 170 & 25 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This is A039755. It represents the 2-Dowling arrangements, corresponding to the Whitney numbers for the $\mathrm{B_n}$ lattices [15]. We have in this case

$$L = [e^x, e^x \sinh x],$$

and the production matrix of L is given by

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 5 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 7 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 9 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The row sums of

$$L = [e^x, e^x \sinh x] = \left[e^x, \frac{e^{2x}-1}{2}\right]$$

give the sequence of Dowling numbers A007405 with e.g.f.

$$e^x e^{e^x \sinh x} = e^x \exp\left(\frac{e^{2x}-1}{2}\right).$$

This sequence begins

$$1, 2, 6, 24, 116, 648, 4088, 28640, \ldots,$$

and has Hankel transform [15]

$$h_n = 2^{\binom{n+1}{2}} \prod_{k=1}^{n} k!$$

The sequence of Dowling numbers has g.f. given by the continued fraction

$$\cfrac{1}{1-2x-\cfrac{2x^2}{1-4x-\cfrac{4x^2}{1-6x-\cfrac{6x^2}{1-8x-\cdots}}}}.$$

This follows from the fact that the Dowling numbers coincide with the first column of the exponential Riordan array

$$L_D = [e^x \exp(e^x \sinh x), e^x \sinh x]$$

which has production matrix

$$\left(\begin{array}{ccccccc} 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 6 & 8 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 8 & 10 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 10 & 12 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Thus the Dowling numbers are the moments of the family of orthogonal polynomials $P_n^D(x)$ with coefficient array

$$L_D^{-1} = \left[\frac{e^{-x}}{\sqrt{1+2x}}, \frac{\ln(1+2x)}{2}\right] = \left[\frac{e^{-x}}{\sqrt{1+2x}}, \ln(\sqrt{1+2x})\right],$$

which satisfy the three-term recurrence

$$P_n^D(x) = (x-2n)P_{n-1}^D(x) - 2(n-1)P_{n-1}^D(x),$$

with $P_0^D(x) = 1$ and $P_1^D(x) = x - 2$.

3 The Riordan array $L = \left[\left(\frac{1+\sin x}{\cos^2 x}\right)^r, \frac{\sin x + 1 - \cos x}{\cos x}\right]$

In this example, we start with $f(x) = \frac{\sin x}{\cos x}$ and then modify the numerator expression by adding $1 - \cos x$. Our modified $f(x)$ is now given by

$$f(x) = \frac{\sin x + 1 - \cos x}{\cos x} = \sec x + \tan x - 1.$$

We note that the numbers with e.g.f. $\sec x + \tan x$, which begin

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, \ldots$$

are called "up-down" numbers (or Euler numbers) A000111. We now let

$$g(x) = (f'(x))^r = \left(\frac{1+\sin x}{\cos^2 x}\right)^r = ((1+\sin x)\sec^2 x)^r.$$

Thus we consider the exponential Riordan array

$$L = [(f'(x))^r, f(x)] = [((1+\sin x)\sec^2 x)^r, \sec x + \tan x - 1].$$

We have

$$L^{-1} = \left[\left(\frac{2}{2+2x+x^2}\right)^r, 2\tan^{-1}(1+x) - \frac{\pi}{2}\right].$$

Note that

$$(2\tan^{-1}(1+x) - \frac{\pi}{2})' = \frac{2}{2+2x+x^2}.$$

Proposition 11. *The matrix*

$$L^{-1} = \left[\left(\frac{2}{2+2x+x^2}\right)^r, 2\tan^{-1}(1+x) - \frac{\pi}{2}\right]$$

is the coefficient array of the family of orthogonal polynomials $P_n^{(r)}(x)$ *which satisfy the following three-term recurrence*

$$P_n^{(r)}(x) = (x - r - n + 1)P_{n-1}^{(r)}(x) - ((n-1)r + \binom{n-1}{2})P_{n-2}^{(r)}(x),$$

with $P_0^{(r)}(x) = 1$, $P_1^{(r)}(x) = x - r$.

Proof. We have $g(x) = ((1+\sin x)\sec^2 x)^r$, and $\sec x + \tan x - 1$. Then using equations (1) and (2) we obtain that the generating function of the production array P_L of L is equal to

$$e^{xy}(r(1+x) + (1+x+x^2/2)y).$$

This implies that P_L has the form

$$\begin{pmatrix} r & 1 & 0 & 0 & 0 & 0 & \cdots \\ r & r+1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2r+1 & r+2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3r+3 & r+3 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4r+6 & r+4 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 5r+10 & r+5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which proves the assertion. □

Corollary 12. *The generating function of the moment sequence associated to the family of orthogonal polynomials* $P_n^{(r)}$ *is given by*

$$((1+\sin x)\sec^2 x)^r = \cfrac{1}{1-rx-\cfrac{rx^2}{1-(r+1)x-\cfrac{(2r+1)x^2}{1-(r+2)x-\cfrac{(3r+3)x^2}{1-(r+3)x-\cfrac{(4r+6)x^2}{1-\cdots}}}}}.$$

Corollary 13. *The Hankel transform of the moment sequence associated to the family of orthogonal polynomials* $P_n^{(r)}$ *is given by*

$$h_n = \prod_{k=0}^{n} k!(r+k/2)^{n-k}.$$

For $r=1$, we get the exponential Riordan array

$$L = \left[\frac{1+\sin x}{\cos^2 x}, \frac{\sin x + 1 - \cos x}{\cos x}\right],$$

which begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 5 & 11 & 6 & 1 & 0 & 0 & \cdots \\ 16 & 45 & 35 & 10 & 1 & 0 & \cdots \\ 61 & 211 & 210 & 85 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The first column is made up of the "shortened" or reduced up-down (or Euler) numbers with e.g.f. $(\sec x + \tan x)'$. This matrix is A147315. This matrix has production matrix

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 6 & 4 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 10 & 5 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 15 & 6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

and hence the shortened up-down numbers have generating function given by the continued fraction

$$\cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{3x^2}{1-3x-\cfrac{6x^2}{1-4x-\cfrac{10x^2}{1-5x-\cdots}}}}},$$

and Hankel transform A154604

$$h_n = \prod_{k=1}^{n} \binom{k+1}{2}^{n-k+1} = \prod_{k=0}^{n} \binom{k+2}{2}^{n-k} = \left(\frac{1}{2}\right)^{\binom{n+1}{2}} \prod_{k=0}^{n} k!(k+1)!$$

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