Generalized Narayana Polynomials, Riordan Arrays and Lattice Paths

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Abstract

We study a family of polynomials in two variables, identifying them as the moments of a two-parameter family of orthogonal polynomials. The coefficient array of these orthogonal polynomials is shown to be an ordinary Riordan array. We express the generating function of the sequence of polynomials under study as a continued fraction, and determine the corresponding Hankel transform. An alternative characterization of the polynomials in terms of a related Riordan array is also given. This Riordan array is associated with Łukasiewicz paths. The special form of the production matrices is exhibited in both cases. This allows us to produce a bijection from a set of coloured Łukasiewicz paths to a set of coloured Motzkin paths. The polynomials studied generalize the notion of Narayana polynomial.

1 Introduction

In this note, we shall be concerned with the family of bivariate polynomials defined by

$$Q_n(r,s) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{j} \binom{n-j}{2(k-j)} a_{k-j}(s) r^k,$$

where

$$a_n(s) = \sum_{k=0}^{n} \binom{2n-k-1}{n-k} \frac{k+0^{n+k}}{n+0^{nk}} s^k.$$

Our main goal is to characterize $Q_n(r,s)$ as the moment sequence of a family of orthogonal polynomials, and in so doing also show that the sequence $Q_n(r,s)$ has Hankel transform $h_n(r,s) = \det(Q_{i+j}(r,s))_{0 \le i,j \le n}$ [26] given by

$$h_n(r,s) = s^n r^{\binom{n+1}{2}}.$$

Examples of these sequences include the shifted Catalan numbers $Q_n(1,1)$, the central binomial coefficients $Q_n(1,2)$, the central Delannoy numbers $Q_n(2,2)$, and many other sequences that are documented in the On Line Encyclopedia of Integer Sequences [31, 32].

We use the production matrix of a particular Riordan array to establish the existence of a family of orthogonal polynomials for which the generalized Narayana polynomials are moments. We find it interesting to look at a closely related Riordan array, again with the generalized Narayana polynomials as elements of the first column, but where now the production matrix is not tri-diagonal. It does however have a structure that is interesting in its own right. The Hankel transform of the row sum sequence of this second Riordan array is also interesting to study.

2 Preliminaries on integer sequences, Riordan arrays and Hankel transforms

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is called the *ordinary generating function* or g.f. of the sequence. a_n is thus the coefficient of x^n in this series. We denote this by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ is the n-th Catalan number A000108. We use the notation $0^n = [x^n]1$ for the sequence $1, 0, 0, 0, \ldots$, A000007. Thus $0^n = [n = 0] = \delta_{n,0} = \binom{0}{n}$. Here, we have used the Iverson bracket notation [19], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false.

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)$ such that $f(\bar{f}(x)) = x$.

For a lower triangular matrix $(a_{n,k})_{n,k\ge 0}$ the row sums give the sequence with general term $\sum_{k=0}^{n} a_{n,k}$.

The *Riordan group* [30, 33], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1x + g_2x^2 + \cdots$ and $f(x) = f_1x + f_2x^2 + \cdots$ where $f_1 \neq 0$ [33]. We assume in addition that $f_1 = 1$ in what follows. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g, f)(h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)'$ is an integer sequence with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbb{Z}^{\mathbb{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbb{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \mapsto (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

Example 1. The so-called *binomial matrix* $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$, and hence as an array coincides with Pascal's triangle. More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

For an invertible matrix L, its production matrix (also called its Stieltjes matrix) [15, 16] is the matrix

$$P_L = L^{-1}\hat{L},$$

where $\hat{L}$ is the matrix L with its first row removed. A Riordan array L is the inverse of the coefficient array of a family of orthogonal polynomials if and only if P_L is tri-diagonal [7, 8]. Necessarily, the Jacobi coefficients of these orthogonal polynomials are then constant. Such types of polynomials have been studied in many contexts, including orthogonal polynomials [34, 18, 1, 2], random walks [23, 20], group theory [13], non-commutative probability [29, 3, 11], and the study of random matrices [14].

An important feature of Riordan arrays is that they have a number of sequence characterizations [12, 21]. The simplest of these is as follows.

Proposition 2. *[21, Theorem 2.1, Theorem 2.2] Let $D = [d_{n,k}]$ be an infinite triangular matrix. Then D is a Riordan array if and only if there exist two sequences $A = [a_0, a_1, a_2, \ldots]$ and $Z = [z_0, z_1, z_2, \ldots]$ with $a_0 \neq 0$, $z_0 \neq 0$ such that*

- $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}, \quad (k, n = 0, 1, \ldots)$
- $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}, \quad (n = 0, 1, \ldots).$

The coefficients $a_0, a_1, a_2, \ldots$ and $z_0, z_1, z_2, \ldots$ are called the A-sequence and the Z-sequence of the Riordan array $L = (g(x), f(x))$, respectively. Letting $A(x)$ be the generating function of the A-sequence and $Z(x)$ be the generating function of the Z-sequence, we have

$$A(x) = \frac{x}{\bar{f}(x)}, \quad Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right). \tag{1}$$

The first column of P_L is then generated by $Z(x)$, while the k-th column is generated by $x^{k-1}A(x)$ (taking the first column to be indexed by 0).

For a given sequence a_n, the Hankel transform [26] of a_n is the sequence $\det(a_{i+j})_{0\leq i,j\leq n}$. The Hankel transforms we shall be interested in will all be related to orthogonal polynomials, which in turn will be related to lattice paths [17, 35]. A lattice path [27] is a sequence of vertices in the integer lattice $\mathbb{Z}^2$. A pair of consecutive vertices is called a step of the path. The height of a step is given by the y-value of the first vertex. A valuation is an integer function on the set of possible steps of $\mathbb{Z}^2 \times \mathbb{Z}^2$. A valuation of a path is the product of the valuations of its steps. We concern ourselves with Motzkin paths, k-coloured Motzkin paths [4] and Łukasiewicz paths [35], which are defined below. A path of length n begins at $(0, 0)$ and ends at $(n, 0)$. The valuations will be independent of the x-coordinates of the points, therefore the x-coordinates are redundant and we can represent a path π by the sequence of its y-coordinates $\pi = (\pi(0)....\pi(n))$.

Definition 3. A path $\pi = (\pi(0)....\pi(n))$ is a Motzkin path [27] if it satisfies the following conditions

1. The elementary steps can be north-east, east and south-east.
2. Steps never go below the x axis.
3. $\pi(0) = (0,0)$ and $\pi(n) = (n,0)$

If the east steps are labelled by k colours we obtain the k-coloured Motzkin paths.

Definition 4. A path $\pi = (\pi(0)....\pi(n))$ is a Łukasiewicz path [27] if it satisfies the following conditions

1. The elementary steps can be north - east and east as those in Motzkin paths.
2. In addition south-east steps from level k can fall to any level above or on the x axis, and are denoted as $\lambda_{k,l}$ and referred to as Łukasiewicz steps, where k is the starting point of the south-east step and l the level where the step ends.
3. Steps never go below the x axis.

Theorem 5. *[27, Theorem 2.3] Let*

$$\mu_n = \sum_{\pi \in \mathbf{M}} v(\pi)$$

where the sum is over the set of Motzkin paths $\pi = (\pi(0)....\pi(n))$ of length n. Here $\pi(j)$ is the level after the j^{th} step, and the valuation of a path is the product of the valuations of its steps $v(\pi) = \prod_{i=1}^{n} v_i$ where

$$v_i = v(\pi(i-1), \pi(i)) = \begin{cases} 1 & \textit{if the } i^{th} \textit{ step rises} \\ \beta_{\pi(i-1)} & \textit{if the } i^{th} \textit{ step is horizontal} \\ \alpha_{\pi(i-1)} & \textit{if the } i^{th} \textit{ step falls} \end{cases}$$

$\beta_{\pi(i-1)}$
level i 1
$\alpha_{\pi(i-1)}$

Then the g.f. of the sequence μ_n is given by

$$\mathbf{M}(x) = \sum_{n=0}^{\infty} \mu_n x^n.$$

A continued fraction expansion of the g.f. is then

$$M(x) = \cfrac{1}{1-\beta_0 x - \cfrac{\alpha_1 x^2}{1-\beta_1 x - \cfrac{\alpha_2 x^2}{1-\beta_2 x - \cfrac{\alpha_3 x^2}{\ddots}}}}.$$

Now, from [25] we have the following result,

Theorem 6. *[24, Theorem 11] Let $(a_n)_{n\geq 0}$ be a sequence of numbers with g.f. $\sum_{n=0}^{\infty} a_n x^n$ expressible in the form*

$$\sum_{n=0}^{\infty} a_n x^n = \cfrac{a_0}{1-\beta_0 x - \cfrac{\alpha_1 x^2}{1-\beta_1 x - \cfrac{\alpha_2 x^2}{\ddots}}}.$$

Then the Hankel determinant $\det_{0\leq i,j\leq n-1}(a_{i+j})$ is given by

$$a_0^n \alpha_1^{n-1}\alpha_2^{n-2}\dots\alpha_{n-2}^2\alpha_{n-1} = a_0^n \prod_{k=1}^{n-1}\alpha_k^{n-k},$$

where the sequences $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 0}$ are the coefficients in the recurrence relation

$$P_n(x) = (x-\beta_n)P_{n-1}(x) - \alpha_n P_{n-2}(x), \quad n = 1, 2, 3, 4, \dots$$

of the family of orthogonal polynomials P_n for which a_n forms the moment sequence.

3 Generalized Narayana polynomials

The sequence $a_n(s) = \sum_{k=0}^{n} \binom{2n-k-1}{n-k} \frac{k+0^{n+k}}{n+0^{nk}} s^k$ has generating function [10]

$$g_s(x) = \frac{1}{1-sxc(x)} = (1, xc(x)) \cdot \frac{1}{1-sx},$$

where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers, and $(1, xc(x))$ is the Riordan array [30] whose k-th column has generating function $(xc(x))^k$ [10]. Since $c(x)$ has continued fraction expansion

$$c(x) = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}},$$

the generating function $g_s(x)$ has expansion

$$g_s(x) = \cfrac{1}{1 - \cfrac{sx}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cdots}}}}.$$

The matrix with general term

$$N_{n,k}(s) = \sum_{j=0}^{k} \binom{n}{j} \binom{n-j}{2(k-j)} a_{k-j}(s)$$

is the generalized Narayana triangle [5] defined by the sequence $a_n(s)$, and the polynomials $Q_n(r,s)$ thus represent generalized Narayana polynomials. The Narayana triangle A001263 with general term

$$\frac{1}{k+1}\binom{n+1}{k}\binom{n}{k}$$

corresponds to $s = 1$. For $s = 2$ we get the number triangle with general term $\binom{n}{k}^2$ A008459.

Proposition 7. *The generating function of the polynomials $Q_n(r,s) = \sum_{k=0}^{n} N_{n,k}(s) r^k$ is given by*

$$g_Q(x) = \cfrac{1}{1-(r+1)x - \cfrac{srx^2}{1-(r+1)x - \cfrac{rx^2}{1-(r+1)x - \cfrac{rx^2}{1-\cdots}}}}.$$

Proof. See Theorem 21, [5]. □

Corollary 8. *The Hankel transform of $Q_n(r,s)$ is given by*

$$h_n = s^n r^{\binom{n+1}{2}}.$$

Proof. This follows from [24, 25]. □

We know [6] that the Narayana polynomials $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k} r^k$ can be characterized as being the first column elements of the inverse Riordan array

$$\left(\frac{1}{1+(r+1)x+rx^2}, \frac{x}{1+(r+1)x+rx^2} \right)^{-1}.$$

The following proposition generalizes this result, by providing a characterization of the polynomials $Q_n(r,s)$ in terms of Riordan arrays.

Proposition 9. *The polynomials $Q_n(r,s)$ are given by the terms of the first column of the Riordan array*

$$L=\left(\frac{1+r(1-s)x^2}{1+(r+1)x+rx^2},\frac{x}{1+(r+1)x+rx^2}\right)^{-1}.$$

Proof. We can express $g_Q(x)$ in closed form by noticing that

$$g_Q(x)=\frac{1}{1-(r+1)x-srx^2v},$$

where v is the solution of the equation

$$v=\frac{1}{1-(r+1)x-rx^2v}.$$

(The function v is the generating function of the Narayana triangle). We find that

$$g_Q(x)=\frac{2}{s\sqrt{x^2(r-1)^2-2x(r+1)+1}+(x(r+1)-1)(s-2)}.$$

Standard Riordan array techniques [7] now show that this is equal to the generating function of the first column of the inverse array $\left(\frac{1+r(1-s)x^2}{1+(r+1)x+rx^2},\frac{x}{1+(r+1)x+rx^2}\right)^{-1}$. ∎

Note that we can also express $g_Q(x)$ as follows.

$$g_Q(x)=\frac{s\sqrt{x^2(r-1)^2-2x(r+1)+1}+(2-s)(x(r+1)-1)}{2(x^2(r^2(s-1)-r(s^2-2s+2)+s-1)+(s-1)(1-2x(r+1)))}.$$

We have

$$L=\left(g_Q(x),\frac{1-x(r+1)-\sqrt{1-2x(r+1)+x^2(r-1)^2}}{2rx}\right).$$

Corollary 10. *The polynomials $Q_n(r,s)$ are the moments of the family of orthogonal polynomials $P_n(x)$ that satisfy the recurrence*

$$P_n(x)=(x-(r+1))P_{n-1}(x)-rP_{n-2}(x),\quad n>2,$$

$P_0(x)=1,\ P_1(x)=x-(r+1),\ P_2(x)=(x-(r+1))P_1(x)-srP_0(x)=x^2-2(r+1)x+r^2+r(2-s)+1.$

Proof. We need to show [7, 8] that the production matrix [15, 16] of

$$L=\left(\frac{1+r(1-s)x^2}{1+(r+1)x+rx^2},\frac{x}{1+(r+1)x+rx^2}\right)^{-1}$$

is tri-diagonal. Standard Riordan array techniques show that this is so, with form

$$P_L = \begin{pmatrix} r+1 & 1 & 0 & 0 & 0 & 0 & \dots \\ sr & r+1 & 1 & 0 & 0 & 0 & \dots \\ 0 & r & r+1 & 1 & 0 & 0 & \dots \\ 0 & 0 & r & r+1 & 1 & 0 & \dots \\ 0 & 0 & 0 & r & r+1 & 1 & \dots \\ 0 & 0 & 0 & 0 & r & r+1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The three-term recurrence coefficients for $P_n(x)$ can be read from P_L. □

Corollary 11. *The Hankel transform of $Q_n(r,s)$ is given by*

$$h_n(r,s) = s^n r^{\binom{n+1}{2}}.$$

Proof. This is a direct consequence of the form of g_Q; equivalently it is derived from the form of the production matrix above. □

We note that $Q_n(r, j/r)$ is an integer sequence for $0 \le j \le r-1$. For instance, $Q_n(2,1/2)$ is the sequence that begins

$$1, 3, 10, 36, 138, 558, 2362, 10398, 47326, 221562, \dots$$

It has generating function

$$\cfrac{1}{1-3x-\cfrac{x^2}{1-3x-\cfrac{2x^2}{1-3x-\cfrac{2x^2}{1-\cdots}}}}.$$

This means [9] that $Q_n(2,1/2)$ is the third binomial transform of the aeration of the sequence with generating function

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{2x}{1-\cdots}}}}.$$

This latter sequence is the sequence of generalized Catalan numbers $C(2;n)$ A064062 with g.f.

$$\frac{1}{1-xc(2x)}.$$

Thus the g.f. of $Q_n(2,1/2)$ can be represented as

$$\frac{1}{1-3x}\frac{1}{1-\left(\frac{x}{1-3x}\right)^2 c\left(2\left(\frac{x}{1-3x}\right)^2\right)}.$$

The Hankel transform of $Q_n(2,1/2)$ is given by $h_n(2,1/2)=\left(\frac{1}{2}\right)^n 2^{\binom{n+1}{2}}=2^{\binom{n}{2}}$, A006125, a sequence with many combinatorial interpretations. One such is that it represents the number of graphs on n labeled nodes.

Similarly the sequence $Q_n(3,2/3)$ which begins

$$1, 4, 18, 88, 458, 2504, 14244, 83696, 505114, 3116968, \ldots,$$

with g.f.

$$\cfrac{1}{1-4x-\cfrac{2x^2}{1-4x-\cfrac{3x^2}{1-4x-\cfrac{3x^2}{1-\cdots}}}},$$

can be seen to be the fourth binomial transform of the aeration of the sequence with g.f.

$$\frac{1}{1-2xc(3x)}.$$

The Hankel transform of $Q_n(3,2/3)$ is given by

$$h_n(3,2/3)=\left(\frac{2}{3}\right)^n 3^{\binom{n+1}{2}},$$

which begins

$$1, 2, 12, 216, 11664, 1889568, 918330048, \ldots.$$

This is A083667, the number of antisymmetric binary relations on a set of n labeled points.

We note that in general, we can interpret the continued fraction form of $g_Q(x)$ as saying that $Q_n(r,s)$ is the $(r+1)$-st binomial transform of the aeration of the sequence with g.f.

$$\frac{1}{1-srxc(rx)}.$$

Thus we may express $g_Q(x)$ as

$$g_Q(x)=\frac{1}{1-(r+1)x}\frac{1}{1-sr\left(\frac{x}{1-(r+1)x}\right)^2 c\left(r\left(\frac{x}{1-(r+1)x}\right)^2\right)}.$$

Other interesting sequences are $Q_n(n,0)=(n+1)^n$, or A000169, $Q_n(n,1)=\sum_{k=0}^n \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k}n^k$, which is A099169, and $Q_n(n,2)=\sum_{k=0}^n \binom{n}{k}^2 n^k$, which is A187021.

4 Moment representation

We have the following moment representation of $Q_n(r,s)$.

$$Q_n(r,s)=\begin{cases}1, \quad r=0\\ (r+1)^n, \quad s=0\\ \\ \sum_{k=0}^n \frac{1}{k+1}\binom{n+1}{k}\binom{n}{k}r^k=\frac{1}{2\pi}\int_{r+1-2\sqrt{r}}^{r+1+2\sqrt{r}} x^n \frac{\sqrt{-x^2+2x(r+1)-(r-1)^2}}{r}\,dx, \quad s=1, r\neq 0,\\ \\ \frac{s}{2\pi}\int_{r+1-2\sqrt{r}}^{r+1+2\sqrt{r}} x^n \frac{\sqrt{-x^2+2x(r+1)-(r-1)^2}}{1-s+r(s^2-2s+2)-r^2(s-1)+2x(r+1)(s-1)-x^2(s-1)}\,dx+\\ \qquad \frac{s-2}{2(s-1)}\left(\left(r+1-s\sqrt{\frac{r}{s-1}}\right)^n+\left(r+1+s\sqrt{\frac{r}{s-1}}\right)^n\right), \quad \text{otherwise.}\end{cases}$$

This density function can be seen as a generalization of the Marčenko-Pastur density [28].

5 A product matrix

The motivation for the work in this section comes from the observation that the Riordan array

$$\left(\frac{1+(c-a)x+(d-b)x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}$$

has production array

$$\begin{pmatrix} c & 1 & 0 & 0 & 0 & 0 & \cdots\\ d & a & 1 & 0 & 0 & 0 & \cdots\\ 0 & b & a & 1 & 0 & 0 & \cdots\\ 0 & 0 & b & a & 1 & 0 & \cdots\\ 0 & 0 & 0 & b & a & 1 & \cdots\\ 0 & 0 & 0 & 0 & b & a & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix},$$

while the Riordan array given by the product

$$\left(\frac{1+(c-a)x+(d-b)x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}\cdot\left(1, \frac{x}{1+tx}\right)$$

has production matrix

$$\begin{pmatrix} c & 1 & 0 & 0 & 0 & 0 & \dots \\ d & a-t & 1 & 0 & 0 & 0 & \dots \\ dt & b & a-t & 1 & 0 & 0 & \dots \\ dt^2 & bt & b & a-t & 1 & 0 & \dots \\ dt^3 & bt^2 & bt & b & a-t & 1 & \dots \\ dt^4 & bt^3 & bt^2 & bt & b & a-t & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In other words, we can show that the product matrix has

$$Z(x)=\frac{c-(ct-d)x}{1-tx}, \quad A(x)=\frac{1-(2t-a)x-(at-b-t^2)x^2}{1-tx}.$$

It is possible to display $Q_n(r,s)$ as the first column of a Riordan array in many ways. The Riordan array in the last section was chosen because its inverse is the coefficient array of the associated family of orthogonal polynomials. Another interesting Riordan array is the array

$$\begin{aligned} \tilde{L} &= \left(\left(1, \frac{x}{1-x}\right)\cdot\left(\frac{1+r(1-s)x^2}{1+(r+1)x+rx^2}, \frac{x}{1+(r+1)x+rx^2}\right)\right)^{-1} \\ &= \left(\frac{1-2x-(r(s-1)-1)x^2}{1+(r-1)x}, \frac{x(1-x)}{1+(r-1)x}\right)^{-1} \\ &= \left(\frac{1+r(1-s)x^2}{1+(r+1)x+rx^2}, \frac{x}{1+(r+1)x+rx^2}\right)^{-1}\cdot\left(1, \frac{x}{1+x}\right) \\ &= L\cdot\left(1, \frac{x}{1+x}\right). \end{aligned}$$

We have

$$\tilde{L}=\left(g_Q(x), \frac{1-x(r-1)-\sqrt{1-2x(r+1)+x^2(r-1)^2}}{2}\right).$$

The matrix $\tilde{L}$ then has the production matrix

$$P_{\tilde{L}}=\begin{pmatrix} r+1 & 1 & 0 & 0 & 0 & 0 & \dots \\ rs & r & 1 & 0 & 0 & 0 & \dots \\ rs & r & r & 1 & 0 & 0 & \dots \\ rs & r & r & r & 1 & 0 & \dots \\ rs & r & r & r & r & 1 & \dots \\ rs & r & r & r & r & r & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Such production matrices are linked to Łukasiewicz paths [22].

Example 12. We take the case $r=2$, $s=\frac{1}{2}$. In this case we obtain the production matrix

$$P_{\tilde{L}} = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 1 & 0 & 0 & \dots \\ 1 & 2 & 2 & 2 & 1 & 0 & \dots \\ 1 & 2 & 2 & 2 & 2 & 1 & \dots \\ 1 & 2 & 2 & 2 & 2 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The associated Riordan array is

$$\left(\frac{1-2x+2x^2}{1+x}, \frac{x(1-x)}{1+x}\right)^{-1} = \left(\frac{3-9x-\sqrt{1-6x+x^2}}{2(1-6x+10x^2)}, \frac{1-x-\sqrt{1-6x+10x^2}}{2}\right).$$

This matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 3 & 1 & 0 & 0 & 0 & 0 & \dots \\ 10 & 5 & 1 & 0 & 0 & 0 & \dots \\ 36 & 22 & 7 & 1 & 0 & 0 & \dots \\ 138 & 96 & 38 & 9 & 1 & 0 & \dots \\ 588 & 426 & 192 & 58 & 11 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and has row sums u_n that begin

$$1, 4, 16, 66, 282, 1246, 5674, 26530, 126910, 619086, \dots,$$

with g.f.

$$\frac{1-3x-2x^2-(1-2x)\sqrt{1-6x+x^2}}{2x(1-6x+10x^2)}.$$

We have the moment representation

$$u_n = \frac{1}{2\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}} x^n \frac{(x-2)\sqrt{-x^2+6x-1}}{x^2-6x+10}\,dx.$$

Note also that the first column sequence $1, 3, 10, 36, \dots$ has the moment representation

$$u_n = \frac{1}{2\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}} x^n \frac{\sqrt{-x^2+6x-1}}{x^2-6x+10}\,dx.$$

The Hankel transform of the row sum sequence u_n is equal to

$$2^{\binom{n}{2}}[x^n]\frac{1-x}{1-x+2x^2}.$$

In general, the rows sums of $\tilde{L}$ can be shown to have Hankel transform

$$r^{\binom{n}{2}}[x^n]\frac{1-x}{1-rsx+rx^2}.$$

Example 13. We consider the case $r=1$, $s=3$. Then

$$\tilde{L}=(1-2x-x^2, x(1-x))^{-1}=\left(\frac{1}{1-2xc(x)-x^2c(x)^2}, xc(x)\right).$$

The row sums u_n of $\tilde{L}$ have g.f.

$$\frac{1}{(1-2xc(x)-x^2c(x)^2)(1-xc(x))}=\frac{1-5x-(1+x)\sqrt{1-4x}}{2x(x^2+8x-2)}.$$

We have the moment representation

$$u_n=\frac{1}{2\pi}\int_0^4 x^n\frac{(1+x)\sqrt{x(4-x)}}{1+8x-2x^2}\,dx+\frac{1}{8+4\sqrt{2}}\left(2-\frac{3}{\sqrt{2}}\right)^n+\frac{1}{8-4\sqrt{2}}\left(2+\frac{3}{\sqrt{2}}\right)^n.$$

The Hankel transform of the row sums is equal to

$$[x^n]\frac{1-x}{1-3x+x^2},$$

or F_{2n+1}. The row sum sequence is A143464, which begins

$$1, 3, 11, 42, 164, 649, 2591, 10408, \ldots.$$

We can also look at sequences with negative values of r and s. For instance, the sequence $Q_n(-1,-1)$ is associated with the production matrix

$$P_{\tilde{L}}=\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & -1 & -1 & 1 & 0 & 0 & \ldots \\ 1 & -1 & -1 & -1 & 1 & 0 & \ldots \\ 1 & -1 & -1 & -1 & -1 & 1 & \ldots \\ 1 & -1 & -1 & -1 & -1 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The sequence $Q_n(-1,-1)$ begins

$$1, 0, 1, 0, 0, 0, 1, 0, -2, 0, 6, 0, -18, 0, 57, 0, -186, 0, 622, \ldots,$$

and represents the aeration of $C(-1;n)$ A064310. The sequence $Q_n(-1,-1)$ has Hankel transform equal to $(-1)^n$.

Example 14. The case $r=-1$, $s=1$ is also linked to the Fibonacci numbers. For this, we find that the row sums are

$$1, 1, -1, -2, 2, 5, -5, -14, 14, 42, -42, -132, 132, 429, \ldots,$$

a sequence of signed doubled Catalan numbers. The Hankel transform in this case is $(-1)^{\binom{n+1}{2}} F_{n+2}$.

Example 15. Similarly, the case $r=-1$, $s=-1$ is also linked to the Fibonacci numbers. The row sums are given by

$$1, 1, 1, 0, 0, 1, 1, -2, -2, 6, 6, -18, -18, 57, 57, -186, -186, 622, \ldots,$$

a sequence of signed doubled generalized Catalan numbers $C(-1;n)$ (A064310), given by $C(-1; \lfloor \frac{n+1}{2} \rfloor)$. The Hankel transform in this case is

$$1, 0, -1, -1, 2, 3, -5, -8, 13, 21, -34, \ldots,$$

or $(-1)^{\binom{n}{2}} F_{n-1}$. The matrix in question is

$$\left(\frac{2}{3-\sqrt{1-4x^2}}, \frac{1+2x-\sqrt{1-4x^2}}{2}\right) = \left(\frac{1-2x-x^2}{1-2x}, \frac{x(1-x)}{1-2x}\right)^{-1}.$$

6 A series reversion

It is interesting to study the reversion of $xg_Q(x)$, that is, to study the appropriate solution of the equation

$$ug_Q(u) = x.$$

We obtain

$$\frac{u(x)}{x} = \frac{s\sqrt{4rx^2(s-1)+1} + 2x(1-s)(r+1) - s + 2}{2(1-x(r+1)(s-2) + (1-s+r(s^2-2s+2)-r^2(s-1))x^2)}.$$

This again is the generating function of a sequence of polynomials in r and s, which begins

$$1, -(r+1), r^2+r(2-s)+1, -r^3+r^2(2s-3)+r(2s-3)-1, r^4+r^3(4-3s)+r^2(2s^2-7s+6)+r(4-3s)+1, \ldots$$

As polynomials in the variable r, these polynomials have coefficient array given by

$$\bar{N}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & -1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2-s & 1 & 0 & 0 & 0 & \ldots \\ -1 & 2s-3 & 2s-3 & -1 & 0 & 0 & \ldots \\ 1 & 4-3s & 2s^2-7s+6 & 4-3s & 1 & 0 & \ldots \\ -1 & 4s-5 & 2(7s-5)-5s^2 & 2(7s-5)-5s^2 & 4s-5 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which apart from signs, is also Pascal-like. Examples for $s = 0 \ldots 5$ of these matrices $\bar{N}(s)$ are given at the end of the last section. The central coefficients $M(s)$

$$1, 2-s, 2s^2-7s+6, -5s^3+24s^2-38s+20, 14s^4-84s^3+188s^2-187s+70, \ldots$$

of this triangle are of interest. As polynomials in s, they can be expressed as

$$M(s) = \sum_{k=0}^{n} \binom{2n}{n-k} \frac{2k+1}{n+k+1}(-1)^{n-k}(s-1)^{n-k} = \sum_{k=0}^{n} \binom{2n-k-1}{n-k} \frac{k+0^{n+k}}{n+0^{nk}}(1-s)^{n-k}(2-s)^k.$$

The g.f. of $M(s)$ is given by

$$g_M(s) = \frac{1}{1+(s-2)xc((1-s)x)} = \frac{s-(s-2)\sqrt{1-(1-s)4x}}{2(1-(s-2)^2x)}.$$

They are the moments of the family of orthogonal polynomials $R_n(x)$ defined by

$$R_n(x) = (x-2(1-s))R_{n-1}(x) - (s-1)^2 R_{n-2}(x), \quad n > 2,$$

with $R_0(x) = 1$, $R_1(x) = x-(s-2)$ and $R_2(x) = x^2 + x(3s-4) + (s-1)(s-2)$. The coefficient array of this family of polynomials is given by the Riordan array

$$\left(\frac{1-sx-(1-s)x^2}{1+2(1-s)x+(s-1)^2x^2}, \frac{x}{1+2(1-s)x+(s-1)^2x^2} \right).$$

The moments $M(s)$ are then the elements of the first column of the inverse of this array. They can be represented as

$$M_n(s) = \frac{1}{\pi} \int_{4(1-s)}^{0} x^n \frac{(s-2)\sqrt{x(4(1-s)-x)}}{2x(x-s^2-4(1-s))}\, dx$$

for $s \neq 1, 2$. The Hankel transform of $M_n(s)$ is given by

$$h_n(s) = (s-2)^n (s-1)^{n^2}.$$

7 A bijection between paths related to P_L and $P_{\tilde{L}}$

Returning to the production matrix introduced above,

$$P_L = \begin{pmatrix} r+1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ sr & r+1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & r & r+1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & r & r+1 & 1 & 0 & \ldots \\ 0 & 0 & 0 & r & r+1 & 1 & \ldots \\ 0 & 0 & 0 & 0 & r & r+1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

we are now interested in these matrix entries and their corresponding Motzkin paths. It is known [17, 22, 35] that the elements of the first column of the Riordan array corresponding to P_L count Motzkin paths returning to the x axis. Motzkin paths related to the matrix P_L above are referred to as $(r+1)$-coloured Motzkin paths, where we have a choice of $(r+1)$ colours for each east step. Similarly, each south-east step has a choice of r colours. We note that for the matrix above, the south-east step returning to the x axis has a choice of rs colours. For this reason, we will refer to such Motzkin weighted paths as *(s-e_{x-axis}, e_{x-axis}; s-e, e)-Motzkin paths.* Thus, P_L corresponds to a $(rs, r+1; r, r+1)$-Motzkin path. Similarly, it can be shown [22] that the first column of the Riordan array corresponding to the production matrix

$$P_{\tilde{L}} = \begin{pmatrix} r+1 & 1 & 0 & 0 & 0 & 0 & \dots \\ rs & r & 1 & 0 & 0 & 0 & \dots \\ rs & r & r & 1 & 0 & 0 & \dots \\ rs & r & r & r & 1 & 0 & \dots \\ rs & r & r & r & r & 1 & \dots \\ rs & r & r & r & r & r & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

counts Łukasiewicz paths returning to the x axis. Adopting similar notation as for the Motzkin paths above, $P_{\tilde{L}}$ corresponds with a $(rs, r+1; r, r)$-Łukasiewicz path. Now, as the first column of the Riordan arrays corresponding to P_L and $P_{\tilde{L}}$ both have generating function $g_Q(x)$, we introduce the following bijection between Motzkin and Łukasiewicz paths.

Proposition 16. *A bijection exists between the set of $(rs, r+1; r, r+1)$-Motzkin paths of length n and the set of $(rs, r+1; r, r)$-Łukasiewicz paths of length n, for any $n > 0$.*

We choose an ordering of colours of the east steps. In the proof below we refer to the east steps that are coloured with the $(r+1)$-th colour as e_{r+1}

Proof. We construct a map $\phi : \mathbb{M} \to \mathbb{L}$. Given a $(rs, r+1; r, r+1)$-Motzkin path P of length n, we can obtain a path $\phi(P)$ as follows:

- i-coloured east steps remain unchanged, for $i < r+1$.
- East steps e_{r+1}, become Łukasiewicz steps, taking the colour of its associated north-east, south-east pair.

Conversely, given a $(rs, r+1; r, r)$-Łukasiewicz path we can obtain a $(rs, r+1; r, r+1)$-Motzkin path by

- i-coloured east steps remain unchanged, for $i < r+1$.
- Łukasiewicz steps form the $(r+1)^{th}$ coloured east steps, e_{r+1}.

We illustrate the Łukasiewicz path formed from the mapping ϕ, for a Motzkin path with all east steps e_{r+1}. We see that each north-east step is paired with its right horizontally visible south-east step. All east steps at the height of the south-east step belong to that north-east, south-east pair. For a north-east, south-east pair, with m level steps, at height $(y+1)$, the Łukasiewicz path becomes a north-east path of $(m+1)$ steps with a Łukasiewicz step $\lambda_{(y+m+1,y)}$. The Łukasiewicz step now takes the choice of colours of the original south-east step. North-east, south-east pairs with no east steps remain unchanged. We note that it is clear from above that the length is preserved.

8 Examples

The table below lists the first six generalized Narayana triangles, in the sense of this article. For $s = 0$, we have the usual Pascal's triangle, A007318. For $s = 1$, we have the Narayana triangle A001263, while for $s = 2$ we have A008459, which has general term $\binom{n}{k}^2$.

$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{array}$	$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 6 & 6 & 1 & 0 & 0 \\ 1 & 10 & 20 & 10 & 1 & 0 \\ 1 & 15 & 50 & 50 & 15 & 1 \end{array}$	$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 1 & 9 & 9 & 1 & 0 & 0 \\ 1 & 16 & 36 & 16 & 1 & 0 \\ 1 & 25 & 100 & 100 & 25 & 1 \end{array}$
$s = 0$	$s = 1$	$s = 2$
$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 1 & 0 & 0 & 0 \\ 1 & 12 & 12 & 1 & 0 & 0 \\ 1 & 22 & 54 & 22 & 1 & 0 \\ 1 & 35 & 160 & 160 & 35 & 1 \end{array}$	$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & 0 & 0 \\ 1 & 15 & 15 & 1 & 0 & 0 \\ 1 & 28 & 74 & 28 & 1 & 0 \\ 1 & 45 & 230 & 230 & 45 & 1 \end{array}$	$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 1 & 0 & 0 & 0 \\ 1 & 18 & 18 & 1 & 0 & 0 \\ 1 & 34 & 96 & 34 & 1 & 0 \\ 1 & 55 & 310 & 310 & 55 & 1 \end{array}$
$s = 3$	$s = 4$	$s = 5$

The next table lists some examples of the sequences $Q_n(r,s)$ for the values of r and s shown.

(r,s)	$Q_n(r,s)$	OEIS number
(0,1)	$1, 1, 1, 1, 1, 1, \ldots$	A000012
(1,1)	$1, 2, 5, 14, 42, 132 \ldots$	A000108$(n+1)$
(2,1)	$1, 3, 11, 45, 197, 903 \ldots$	A001003$(n+1)$
(3,1)	$1, 4, 19, 100, 562, 3304 \ldots$	A007564$(n+1)$
(4,1)	$1, 5, 29, 185, 1257, 8925 \ldots$	A059231$(n+1)$
(0,2)	$1, 1, 1, 1, 1, 1, \ldots$	A000012
(1,2)	$1, 2, 6, 20, 70, 252 \ldots$	A000984
(2,2)	$1, 3, 13, 63, 321, 1683 \ldots$	A001850
(3,2)	$1, 4, 22, 136, 886, 5944 \ldots$	A069835
(4,2)	$1, 5, 33, 245, 1921, 15525 \ldots$	A084771
(1,3)	$1, 2, 7, 26, 100, 392 \ldots$	A101850
(1,4)	$1, 2, 8, 32, 132, 552 \ldots$	A155084

Finally, we document the start of the matrices $\bar{N}(s)$ for $s = 0 \ldots 5$.

1	0	0	0	0	0
−1	−1	0	0	0	0
1	2	1	0	0	0
−1	−3	−3	1	0	0
1	4	6	4	1	0
−1	−5	−10	−10	−5	−1

$s = 0$

1	0	0	0	0	0
−1	−1	0	0	0	0
1	1	1	0	0	0
−1	−1	−1	−1	0	0
1	1	1	1	1	0
−1	−1	−1	−1	−1	−1

$s = 1$

1	0	0	0	0	0
−1	−1	0	0	0	0
1	0	1	0	0	0
−1	1	1	−1	0	0
1	−2	0	−2	1	0
−1	3	−2	−2	3	−1

$s = 2$

1	0	0	0	0	0
−1	−1	0	0	0	0
1	−1	1	0	0	0
−1	3	3	−1	0	0
1	−5	3	−5	1	0
−1	7	−13	−13	7	−1

$s = 3$

1	0	0	0	0	0
−1	−1	0	0	0	0
1	−2	1	0	0	0
−1	5	5	−1	0	0
1	−8	10	−8	1	0
−1	11	−34	−34	11	−1

$s = 4$

1	0	0	0	0	0
−1	−1	0	0	0	0
1	−3	1	0	0	0
−1	7	7	−1	0	0
1	−11	21	−11	1	0
−1	15	−65	−65	15	−1

$s = 5$

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