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Continued Fractions and Transformations of Integer Sequences

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Abstract

We show how various transformations of integer sequences, normally realized by Riordan or generalized Riordan arrays, can be translated into continued fraction form. We also examine the Deleham number triangle construction using bi-variate continued fractions, giving examples from the field of associahedra.

1 Introduction

Many classical sequences have generating functions with well known representations as continued fractions. Many other important sequences arise from applying transformations to such sequences with known continued fraction representations. Thus if we can represent the result of the transformation in continued fraction form, we can infer the continued fraction representation of the new sequence.

The transformations that we shall discuss in this note will all be described by (ordinary) Riordan arrays, or generalized Riordan arrays. Thus we shall devote the next section to an overview of the Riordan group.

Sequences will be referred to by their *Annnnnn* number, as found in the On-Line Encyclopedia of Integer Sequences [10, 11].

The reader is referred to [13] for a general reference on continued fractions.

Example 1. The Catalan numbers. The Catalan numbers A000108

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

have generating function

$$C(x) = \frac{1-\sqrt{1-4x}}{2x},$$

which can be represented as, for instance,

$$C(x) = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}},$$

or as

$$C(x) = \cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-\cdots}}}}.$$

We shall use this notation of $C(x)$ for the generating function of the Catalan numbers and c_n for the n-th Catalan number throughout.
Similarly, the g.f. of the central binomial coefficients $\binom{2n}{n}$ A000984, $\frac{1}{\sqrt{1-4x}}$, may be represented as

$$\frac{1}{\sqrt{1-4x}} = \cfrac{1}{1-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}}}$$

or as

$$\frac{1}{\sqrt{1-4x}} = \cfrac{1}{1-2x-\cfrac{2x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-\cdots}}}}.$$

In the sequel we will have occasion to use the Iverson bracket notation [5], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false. For instance, $\delta_{ij} = [i = j]$, while $\delta_n = [n = 0]$.

Note also that if we have a sequence $a_0, a_1, a_2, \ldots$ then the *aeration* of this sequence is the sequence $a_0, 0, a_1, 0, a_2, 0, a_3, 0, \ldots$ with interpolated zeros. If a_n has generating function $g(x)$, then the aerated sequence has generating function $g(x^2)$.

2 Riordan group

The *Riordan group* [9, 12], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1x + g_2x^2 + \ldots$ and $f(x) =$

$f_1 x + f_2 x^2 + \dots$ where $f_1 \neq 0$ [12]. The associated matrix is the matrix whose j-th column is generated by $g(x)f(x)^j$ (the first column being indexed by 0). Thus the i-th element of the j-th column is given by

$$T_{i,j} = [x^i]g(x)f(x)^j$$

where the operator $[x^n]$ [7] extracts the coefficient of x^n from the power series that it is applied to. The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) * (h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.
A Riordan array of the form $(g(x), x)$, where $g(x) = \sum_{k=0}^{n} a_k x^k$ is the generating function of the sequence a_n, is called the *sequence array* of the sequence a_n. Its general term is

$$T_{n,k} = [x^n]g(x)x^k = [x^{n-k}]g(x) = a_{n-k}.$$

Arrays of this form constitute a subgroup of the Riordan group, called the *Appell* group [6].

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \dots)^\top$ is an integer sequence with ordinary generating function $\mathcal{A}\ (x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. This follows since if $\mathbf{M} = (T_{n,k})_{n,k \geq 0}$, we have

$$\begin{aligned}
\sum_{k=0}^{n} T_{n,k} a_k &= \sum_{k=0}^{\infty} [x^n] g(x) f(x)^k a_k \\
&= [x^n] g(x) \sum_{k=0}^{\infty} f(x)^k a_k \\
&= [x^n] g(x) \mathcal{A}(f(x)).
\end{aligned}$$

The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbf{Z}^{\mathbf{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbf{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \longrightarrow (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

Example 2. The binomial matrix $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$. More generally, $\mathbf{B}^r$ is the element $(\frac{1}{1-rx}, \frac{x}{1-rx})$ of the Riordan group, with general term $\binom{n}{k} r^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-r}$ of $\mathbf{B}^r$ is given by $(\frac{1}{1+rx}, \frac{x}{1+rx})$.

If $f_1 = 0$ we call the matrix a "generalized" Riordan array. Such a matrix is not invertible.

3 The Binomial transform $b_n = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} a_k$.

A common transformation on integer sequences is the so-called "binomial transform", which maps the sequence with general term a_n to the sequence with general term b_n defined by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

More generally, for $r \in \mathbb{Z}$, we can define the "r-th binomial transform" of a_n to be the sequence with general term

$$b_n = \sum_{k=0}^{n} r^{n-k} \binom{n}{k} a_k.$$

The theory of Riordan arrays now tells us that this transformation can be represented by the matrix

$$\left(\frac{1}{1-rx}, \frac{x}{1-rx}\right).$$

We recall that if $g(x)$ is the g.f. of the sequence a_n, then the g.f. of the sequence b_n will be given by

$$\left(\frac{1}{1-rx}, \frac{x}{1-rx}\right) \cdot g(x) = \frac{1}{1-rx} g\left(\frac{x}{1-rx}\right).$$

Applying this to the continued fraction representations above, we obtain the following expressions for the generating function of the r-th binomial transform of the Catalan numbers. Firstly

$$\begin{aligned}
\left(\frac{1}{1-rx}, \frac{x}{1-rx}\right) \cdot C(x) &= \frac{1}{1-rx} \cfrac{1}{1-\cfrac{\frac{x}{1-rx}}{1-\cfrac{\frac{x}{1-rx}}{1-\cfrac{\frac{x}{1-rx}}{1-\cdots}}}} \\
&= \cfrac{1}{1-rx-\cfrac{x}{1-\cfrac{\frac{x}{1-rx}}{1-\cfrac{\frac{x}{1-rx}}{1-\cdots}}}} \\
&= \cfrac{1}{1-rx-\cfrac{x}{1-\cfrac{x}{1-rx-\cfrac{x}{1-\cfrac{x}{1-rx-\cfrac{x}{1-\cdots}}}}}}
\end{aligned}$$

and secondly,

$$\left(\frac{1}{1-rx}, \frac{x}{1-rx}\right)\cdot C(x) = \frac{1}{1-rx}\cfrac{1}{1-\frac{x}{1-rx}-\cfrac{\frac{x^2}{(1-rx)^2}}{1-2\frac{x}{1-rx}-\cfrac{\frac{x^2}{(1-rx)^2}}{1-2\frac{x}{1-rx}-\cfrac{\frac{x^2}{(1-rx)^2}}{1-\cdots}}}}$$

$$= \cfrac{1}{1-rx-x-\cfrac{\frac{x^2}{1-rx}}{1-2\frac{x}{1-rx}-\cfrac{\frac{x^2}{(1-rx)^2}}{1-2\frac{x}{1-rx}-\cfrac{\frac{x^2}{(1-rx)^2}}{1-\cdots}}}}$$

$$= \cfrac{1}{1-rx-x-\cfrac{x^2}{1-rx-2x-\cfrac{x^2}{1-rx-2x-\cfrac{x^2}{1-\cdots}}}}.$$

Generalizing this example in an obvious way, we obtain the following two propositions.

Proposition 3. *Let a_n be a sequence with generating function $g(x)$ expressible in the form*

$$g(x) = \cfrac{1}{1-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{1-\cdots}}}.$$

Then the r-th binomial transform of a_n has generating function given by

$$\cfrac{1}{1-rx-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{1-rx-\cfrac{\alpha_3 x}{1-\cfrac{\alpha_4 x}{1-rx-\cfrac{\alpha_5 x}{1-\cdots}}}}}}.$$

Proposition 4. *Let a_n be a sequence with generating function $g(x)$ expressible in the form*

$$g(x) = \cfrac{1}{1-\alpha_1 x-\cfrac{\beta_1 x^2}{1-\alpha_2 x-\cfrac{\beta_2 x^2}{1-\cdots}}}.$$

Then the r-th binomial transform of a_n has generating function given by

$$\cfrac{1}{1-rx-\alpha_1 x-\cfrac{\beta_1 x^2}{1-rx-\alpha_2 x-\cfrac{\beta_2 x^2}{1-rx-\alpha_3 x-\cfrac{\beta_3 x^2}{1-rx-\cdots}}}}.$$

We note that the last expression may be written as

$$\cfrac{1}{1-(\alpha_1+r)x-\cfrac{\beta_1 x^2}{1-(\alpha_2+r)x-\cfrac{\beta_2 x^2}{1-(\alpha_3+r)x-\cfrac{\beta_3 x^2}{1-rx-\cdots}}}}.$$

Thus the form of the continued fraction in this case does not change: only the coefficient of x in each case is incremented (or decremented). It can be shown that through the mechanism of series reversion, this translates the fact that

$$\left(\frac{1}{1-\alpha x-\beta x^2},\frac{x}{1-\alpha x-\beta x^2}\right)\cdot\left(\frac{1}{1-rx},\frac{x}{1-rx}\right)=$$
$$\left(\frac{1}{1-(\alpha+r)x-\beta x^2},\frac{x}{1-(\alpha+r)x-\beta x^2}\right).$$

Example 5. The central trinomial coefficients. The central binomial coefficients $\binom{2n}{n}$ have g.f. expressible as

$$\cfrac{1}{1-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}}}.$$

Thus the central trinomial coefficients A002426, which are the (first) inverse binomial transform of $\binom{2n}{n}$, have g.f expressible as

$$\cfrac{1}{1+x-\cfrac{2x}{1-\cfrac{x}{1+x-\cfrac{x}{1-\cfrac{x}{1+x-\cfrac{x}{1-\cdots}}}}}}.$$

Example 6. The Motzkin numbers. The Motzkin numbers M_n A001006 are given by the binomial transform of the "aerated" Catalan numbers $1, 0, 1, 0, 2, 0, 5, \ldots$, which have g.f. $C(x^2)$. Now

$$C(x^2) = \cfrac{1}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{1 - \cdots}}}$$

and so the Motzkin numbers have g.f. given by

$$M(x) = \cfrac{1}{1 - x - \cfrac{x^2}{1 - x - \cfrac{x^2}{1 - x - \cfrac{x^2}{1 - \cdots}}}}.$$

Example 7. The central trinomial coefficients (revisited). The central trinomial coefficients may also be expressed as the binomial transform of the aerated central binomial coefficients. These latter have g.f. expressible as

$$\frac{1}{\sqrt{1-4x^2}} = \cfrac{1}{1 - \cfrac{2x^2}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{1 - \cdots}}}}$$

and thus the central trinomial coefficients have g.f. expressible as

$$\cfrac{1}{1 - x - \cfrac{2x^2}{1 - x - \cfrac{x^2}{1 - x - \cfrac{x^2}{1 - \cdots}}}}.$$

4 The transformation $b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} r^{n-2k} a_k$.

The matrix that has general term

$$\left[k \le \lfloor \frac{n}{2} \rfloor\right] \binom{n-k}{k} r^{n-2k}$$

can be represented by the generalized Riordan array

$$\left(\frac{1}{1-rx}, \frac{x^2}{1-rx}\right).$$

This follows since the general term of

$$\left(\frac{1}{1-rx}, \frac{x^2}{1-rx}\right)$$

is given by

$$\begin{aligned}
[x^n]\left(\frac{1}{1-rx}\right)\left(\frac{x^{2k}}{(1-rx)^k}\right) &= [x^{n-2k}](1-rx)^{-k-1} \\
&= [x^{n-2k}]\sum_{j=0}^{\infty}\binom{-k-1}{j}(-1)^j r^j x^j \\
&= [x^{n-2k}]\sum_{j=0}^{\infty}\binom{n+k}{j} r^j x^j \\
&= [n-2k\geq 0]\binom{n-k}{n-2k} r^{n-2k}.
\end{aligned}$$

We thus have

Proposition 8. *Let a_n be a sequence with generating function expressible in the form*

$$g(x) = \cfrac{1}{1-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{1-\cdots}}}.$$

Then the sequence b_n with general term

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-k}{k} r^{n-2k} a_k$$

has g.f. expressible as

$$\cfrac{1}{1-rx-\cfrac{\alpha_1 x^2}{1-\cfrac{\alpha_2 x^2}{1-rx-\cfrac{\alpha_3 x^2}{1-\cfrac{\alpha_4 x^2}{1-\cdots}}}}}.$$

Proof. The g.f. of the transformed sequence is given by

$$\left(\frac{1}{1-rx}, \frac{x^2}{1-rx}\right) \cdot g(x) = \frac{1}{1-x} \cfrac{1}{1-\cfrac{\alpha_1 \frac{x^2}{1-rx}}{1-\cfrac{\alpha_2 \frac{x^2}{1-rx}}{1-\cdots}}}$$

$$= \cfrac{1}{1-rx-\cfrac{\alpha_1 x^2}{1-\cfrac{\alpha_2 \frac{x^2}{1-rx}}{1-\cdots}}}$$

$$= \cfrac{1}{1-rx\cfrac{\alpha_1 x^2}{1-\cfrac{\alpha_2 x^2}{1-rx-\cfrac{\alpha_3 x^2}{1-\cfrac{\alpha_4 x^2}{1-rx-\cdots}}}}}.$$

□

Example 9. Number of Motzkin paths of length n with no level steps at odd level. The sequence A090344 with general term

$$a_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} c_k$$

has generating function given by

$$\cfrac{1}{1-x\cfrac{x^2}{1-\cfrac{x^2}{1-x-\cfrac{x^2}{1-\cfrac{x^2}{1-\cdots}}}}}.$$

Proposition 10. *Let a_n be a sequence with generating function $g(x)$ expressible in the form*

$$g(x) = \cfrac{1}{1-\alpha_1 x-\cfrac{\beta_1 x^2}{1-\alpha_2 x-\cfrac{\beta_2 x^2}{1-\cdots}}}.$$

Then the g.f. of the sequence

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} r^{n-2k} a_k$$

is given by

$$\cfrac{1}{1-rx-\alpha_1 x^2-\cfrac{\beta_1 x^4}{1-rx-\alpha_2 x^2-\cfrac{\beta_2 x^4}{1-\cdots}}}.$$

Proof. We have

$$\begin{aligned}\left(\frac{1}{1-rx},\frac{x^2}{1-rx}\right)\cdot g(x) &= \frac{1}{1-rx}\cfrac{1}{1-\alpha_1\frac{x^2}{1-rx}-\cfrac{\beta_1\frac{x^4}{(1-rx)^2}}{1-\alpha_2\frac{x^2}{1-rx}-\cfrac{\beta_2\frac{x^4}{(1-rx)^2}}{1-\cdots}}}\\ &= \cfrac{1}{1-rx-\alpha_1 x^2-\cfrac{\beta_1\frac{x^4}{1-rx}}{1-\alpha_2\frac{x^2}{1-rx}-\cfrac{\beta_2\frac{x^4}{(1-rx)^2}}{1-\cdots}}}\\ &= \cfrac{1}{1-rx-\alpha_1 x^2-\cfrac{\beta_1 x^4}{1-rx-\alpha_2 x^2-\cfrac{\beta_2 x^4}{1-\cdots}}}.\end{aligned}$$

□

Example 11. Number of ordered trees with n edges and having no branches of length 1. This is given by A026418, which begins $1, 0, 1, 1, 2, 3, 6, 11, 22, \ldots$ Now the sequence which begins $1, 1, 2, 3, 6, \ldots$ has general term

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} r^{n-2k} M_k$$

and therefore has g.f. given by

$$\cfrac{1}{1-x-x^2-\cfrac{x^4}{1-x-x^2-\cfrac{x^4}{1-\cdots}}}.$$

5 The transformation $b_n = \sum_{k=0}^{n} \binom{n+k}{2k} a_k$.

The transformation which maps the sequence with general term a_n to the sequence with general term

$$b_n = \sum_{k=0}^{n} \binom{n+k}{2k} a_k$$

can be represented by the Riordan array

$$\left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right).$$

We then have the following proposition.

Proposition 12. *Let a_n be a sequence with generating function $g(x)$ expressible in the form*

$$g(x) = \cfrac{1}{1-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{1-\cdots}}}.$$

Then the sequence with general term b_n given by

$$b_n = \sum_{k=0}^{n} \binom{n+k}{2k} a_k$$

has g.f. expressible as

$$\left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right) \cdot g(x) = \cfrac{1}{1-x-\cfrac{\alpha_1 x}{1-x-\cfrac{\alpha_2 x}{1-x-\cdots}}}.$$

Proof. We have

$$\begin{aligned}\left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right) \cdot g(x) &= \frac{1}{1-x}\cfrac{1}{1-\cfrac{\alpha_1 \frac{x}{(1-x)^2}}{1-\cfrac{\alpha_2 \frac{x}{(1-x)^2}}{1-\cdots}}} \\ &= \cfrac{1}{1-x-\cfrac{\alpha_1 \frac{x}{1-x}}{1-\cfrac{\alpha_2 \frac{x}{(1-x)^2}}{1-\cdots}}} \\ &= \cfrac{1}{1-x-\cfrac{\alpha_1 x}{1-x-\cfrac{\alpha_2 x}{1-x-\cdots}}}.\end{aligned}$$

□

Example 13. The large Schröder numbers. The large Schröder numbers S_n A006318 can be defined by

$$S_n = \sum_{k=0}^{n} \binom{n+k}{2k} c_k.$$

Thus they have g.f. expressible as

$$S(x) = \cfrac{1}{1-x-\cfrac{x}{1-x-\cfrac{x}{1-x-\cfrac{x}{1-x-\cdots}}}}.$$

Example 14. The central Delannoy numbers. The central Delannoy numbers d_n A001850 can be defined by

$$d_n = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}.$$

Thus they have g.f. expressible as

$$\cfrac{1}{1-x-\cfrac{2x}{1-x-\cfrac{x}{1-x-\cfrac{x}{1-x-\cdots}}}}.$$

More generally, we can consider the action of the Riordan array

$$\left(\frac{1}{1-rx}, \frac{x}{(1-rx)^2}\right)$$

for $r \in \mathbb{Z}$. We obtain

Proposition 15. *Let a_n be a sequence with generating function $g(x)$ expressible in the form*

$$g(x) = \cfrac{1}{1-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{1-\cdots}}}.$$

Then the sequence b_n with g.f. given by

$$\frac{1}{1-rx} g\left(\frac{x}{(1-rx)^2}\right)$$

is the expansion of

$$\cfrac{1}{1-rx-\cfrac{\alpha_1 x}{1-rx-\cfrac{\alpha_2 x}{1-rx-\cfrac{\alpha_3 x}{1-rx-\cdots}}}}.$$

In this case,

$$b_n=\sum_{k=0}^{n}\binom{n+k}{2k}r^{n-k}a_k$$

is the general element of the transformed sequence.

Example 16. The case $r=-1$. This case corresponds to the Riordan array

$$\left(\frac{1}{1+x},\frac{x}{(1+x)^2}\right).$$

Now we have, for instance,

$$\sum_{k=0}^{n}\binom{n+k}{2k}(-1)^{n-k}c_k=0^n=\delta_{0n},$$

where 0^n is the sequence $1,0,0,0,\ldots$ with g.f. equal to 1. Thus we obtain the identity

$$1=\cfrac{1}{1+x-\cfrac{x}{1+x-\cfrac{x}{1+x-\cfrac{x}{1+x-\cdots}}}}. \tag{1}$$

Similarly the identity

$$\sum_{k=0}^{n}\binom{n+k}{2k}(-1)^{n-k}\binom{2k}{k}=1$$

yields the identity

$$\frac{1}{1-x}=\cfrac{1}{1+x-\cfrac{2x}{1+x-\cfrac{x}{1+x-\cfrac{x}{1+x-\cdots}}}}. \tag{2}$$

Proposition 17. *Let a_n be a sequence with generating function $g(x)$ expressible in the form*

$$g(x)=\cfrac{1}{1-\alpha_1 x-\cfrac{\beta_1 x^2}{1-\alpha_2 x-\cfrac{\beta_2 x^2}{1-\cdots}}}.$$

Then the generating function of the sequence

$$b_n = \sum_{k=0}^{n} \binom{n+k}{2k} r^{n-k} a_k,$$

obtained by applying the transformation represented by

$$\left(\frac{1}{1-rx}, \frac{x}{(1-rx)^2} \right)$$

to a_n has g.f. given by

$$\cfrac{1-rx}{(1-rx)^2 - \alpha_1 x - \cfrac{\beta_1 x^2}{(1-rx)^2 - \alpha_2 x - \cfrac{\beta_2 x^2}{(1-rx)^2 - \cdots}}}.$$

Proof. The result follows by considering the expression

$$\frac{1-rx}{(1-rx)^2} \cfrac{1}{1 - \alpha_1 \frac{x}{(1-rx)^2} - \cfrac{\beta_1 \frac{x^2}{(1-rx)^4}}{1 - \alpha_2 \frac{x}{(1-rx)^2} - \cfrac{\beta_2 \frac{x^2}{(1-rx)^4}}{1 - \cdots}}}.$$

□

Example 18. Large Schröder numbers (revisited). Since the large Schröder numbers are given by

$$S_n = \sum_{k=0}^{n} \binom{n+k}{2k} c_k$$

we obtain the following expression for their generating function :

$$S(x) = \cfrac{1-x}{(1-x)^2 - x - \cfrac{x^2}{(1-x)^2 - 2x - \cfrac{x^2}{(1-x)^2 - 2x - \cfrac{x^2}{1 - \cdots}}}}$$

or

$$S(x) = \cfrac{1-x}{1 - 3x + x^2 - \cfrac{x^2}{1 - 4x + x^2 - \cfrac{x^2}{1 - 4x + x^2 - \cfrac{x^2}{1 - \cdots}}}}.$$

Note that the above example shows that the partial sums of the large Schröder numbers, which have g.f. given by $\frac{1}{1-x}S(x)$, can be obtained from the expansion of

$$\frac{1}{1-x}S(x) = \cfrac{1}{(1-x)^2 - x - \cfrac{x^2}{(1-x)^2 - 2x - \cfrac{x^2}{(1-x)^2 - 2x - \cfrac{x^2}{1-\cdots}}}}. \tag{3}$$

This can be put in a more general context by observing that the general term of the matrix

$$\left(\frac{1}{(1-rx)^2}, \frac{x}{(1-rx)^2}\right)$$

is given by

$$\binom{n+k+1}{n-k} r^{n-k} = \binom{n+k+1}{2k+1} r^{n-k}.$$

Thus we have the proposition

Proposition 19. *Let a_n be a sequence with generating function $g(x)$ expressible in the form*

$$g(x) = \cfrac{1}{1 - \alpha_1 x - \cfrac{\beta_1 x^2}{1 - \alpha_2 x - \cfrac{\beta_2 x^2}{1-\cdots}}}.$$

Then the generating function of the sequence

$$b_n = \sum_{k=0}^{n} \binom{n+k+1}{2k+1} r^{n-k} a_k,$$

obtained by applying the transformation represented by

$$\left(\frac{1}{(1-rx)^2}, \frac{x}{(1-rx)^2}\right)$$

to a_n has g.f. given by

$$\cfrac{1}{(1-rx)^2 - \alpha_1 x - \cfrac{\beta_1 x^2}{(1-rx)^2 - \alpha_2 x - \cfrac{\beta_2 x^2}{(1-rx)^2 - \cdots}}}.$$

For instance, the g.f. of the partial sums of the large Schröder numbers, which have general term

$$\sum_{k=0}^{n} \binom{n+k+1}{2k+1} c_k,$$

can be represented as above in Eq. (3). This sequence is A086616.

In a similar fashion, we have the result.

Proposition 20. *Let a_n be a sequence with generating function $g(x)$ expressible in the form*

$$g(x) = \cfrac{1}{1 - \alpha_1 x - \cfrac{\beta_1 x^2}{1 - \alpha_2 x - \cfrac{\beta_2 x^2}{1 - \cdots}}}.$$

Then the generating function of the sequence

$$b_n = 0^n + \sum_{k=0}^{n-1} \binom{n+k+1}{2k+1} r^{n-k-1} a_{k+1},$$

obtained by applying the transformation represented by

$$\left(1, \frac{x}{(1-rx)^2}\right)$$

to a_n has g.f. given by

$$\cfrac{(1-rx)^2}{(1-rx)^2 - \alpha_1 x - \cfrac{\beta_1 x^2}{(1-rx)^2 - \alpha_2 x - \cfrac{\beta_2 x^2}{(1-rx)^2 - \cdots}}}.$$

Example 21. Royal paths in a lattice A006319. This is the sequence $1, 1, 4, 16, 68, 304, 1412, \ldots$, which also gives the number of peaks at level 1 in all Schröder paths of semi-length n, $(n \geq 1)$. It has general element

$$0^n + \sum_{k=0}^{n-1} \binom{n+k+1}{2k+1} C_{k+1},$$

and g.f.

$$\cfrac{(1-x)^2}{(1-x)^2 - x - \cfrac{x^2}{(1-x)^2 - 2x - \cfrac{x^2}{(1-x)^2 - 2x - \cfrac{x^2}{(1-x)^2 - 2x - \cdots}}}}.$$

We note similarly that the sequence with g.f. given by

$$\cfrac{(1-x)^2}{(1-x)^2 - x - \cfrac{x^2}{(1-x)^2 - x - \cfrac{x^2}{(1-x)^2 - x - \cfrac{x^2}{(1-x)^2 - x - \cdots}}}}$$

is the transformation of the Motzkin numbers M_n with general element

$$0^n + \sum_{k=0}^{n-1} \binom{n+k+1}{2k+1} M_{k+1}.$$

Finally, the sequence with g.f.

$$\cfrac{(1-x)^2}{(1-x)^2 - 2x - \cfrac{x^2}{(1-x)^2 - 2x - \cfrac{x^2}{(1-x)^2 - 2x - \cfrac{x^2}{(1-x)^2 - 2x - \cdots}}}}$$

is the transformation of the shifted Catalan numbers c_{n+1} with general term

$$0^n + \sum_{k=0}^{n-1} \binom{n+k+1}{2k+1} c_{k+2}.$$

6 The transformation $b_n = \sum_{k=0}^{n} \binom{n}{2k} a_k$.

The matrix with general element $\binom{n}{2k}$ is the generalized Riordan array

$$\left(\frac{1}{1-x}, \frac{x^2}{(1-x)^2} \right).$$

Proposition 22. *Let a_n be a sequence with generating function expressible in the form*

$$g(x) = \cfrac{1}{1 - \cfrac{\alpha_1 x}{1 - \cfrac{\alpha_2 x}{1 - \cdots}}}.$$

Then the sequence with general term

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} a_k$$

has g.f. expressible in the form

$$\cfrac{1}{1 - x - \cfrac{\alpha_1 x^2}{1 - x - \cfrac{\alpha_2 x^2}{1 - x - \cdots}}}.$$

Proof. We have

$$\begin{aligned}
\frac{1}{1-x} g\left(\frac{x^2}{(1-x)^2}\right) &= \frac{1}{1-x} \cfrac{1}{1-\cfrac{\alpha_1 \frac{x^2}{(1-x)^2}}{1-\cfrac{\alpha_2 \frac{x^2}{(1-x)^2}}{1-\cdots}}} \\
&= \cfrac{1}{1-x-\cfrac{\alpha_1 \frac{x^2}{1-x}}{1-\cfrac{\alpha_2 \frac{x^2}{(1-x)^2}}{1-\cdots}}} \\
&= \cfrac{1}{1-x-\cfrac{\alpha_1 x^2}{1-x-\cfrac{\alpha_2 x^2}{1-x-\cdots}}}.
\end{aligned}$$

□

In similar fashion, we can show that for a_n as above, the sequence

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} r^{n-2k} a_k$$

has g.f. expressible in continued fraction form as

$$\cfrac{1}{1-rx-\cfrac{\alpha_1 x^2}{1-rx-\cfrac{\alpha_2 x^2}{1-rx-\cdots}}}.$$

We also have the following:

Proposition 23. *Let a_n be a sequence with generating function $g(x)$ expressible in the form*

$$g(x) = \cfrac{1}{1-\alpha_1 x-\cfrac{\beta_1 x^2}{1-\alpha_2 x-\cfrac{\beta_2 x^2}{1-\cdots}}}.$$

Then the generating function of the sequence

$$b_n = \sum_{k=0}^{n} \binom{n}{2k} a_k$$

is expressible in the form

$$\cfrac{1-x}{(1-x)^2-\alpha_1 x^2-\cfrac{\beta_1 x^4}{(1-x)^2-\alpha_2 x^2-\cfrac{\beta_2 x^4}{(1-x)^2-\alpha_3 x^2-\cdots}}}.$$

Example 24. The Motzkin numbers M_n. We have

$$M_n=\sum_{k=0}^{n}\binom{n}{2k}c_k.$$

Hence the Motzkin numbers have g.f. expressible as

$$\begin{aligned} M(x) &= \cfrac{1-x}{(1-x)^2-x^2-\cfrac{x^4}{(1-x)^2-2x^2-\cfrac{x^4}{(1-x)^2-2x^2-\cdots}}} \\ &= \cfrac{1-x}{1-2x-\cfrac{x^4}{1-2x-x^2-\cfrac{x^4}{1-2x-x^2-\cfrac{x^4}{1-\cdots}}}}. \end{aligned}$$

A simple consequence of this is the fact that the partial sums of the Motzkin numbers, A086615, have generating function expressible as

$$\cfrac{1}{1-2x-\cfrac{x^4}{1-2x-x^2-\cfrac{x^4}{1-2x-x^2-\cfrac{x^4}{1-\cdots}}}}.$$

7 The transformation $b_n=\sum_{k=0}^{n}\binom{n-k}{2k}a_k$.

We consider the generalized Riordan array

$$\left(\frac{1}{1-x},\frac{x^3}{(1-x)^2}\right).$$

The general term of this matrix is given by

$$\begin{aligned} T_{n,k} &= [x^n]\frac{x^{3k}}{(1-x)^{2k+1}} \\ &= [x^{n-3k}]\sum_{i=0}\binom{-2k-1}{i}(-1)^i x^i \\ &= [x^{n-3k}]\sum_{i=0}\binom{2k+1+i-1}{i}x^i \\ &= \binom{2k+n-3k}{n-3k} \\ &= \binom{n-k}{2k}. \end{aligned}$$

We then have

Proposition 25. *Let a_n be a sequence with generating function $g(x)$ expressible in the form*

$$g(x) = \cfrac{1}{1-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{1-\cdots}}}.$$

Then the sequence

$$b_n = \sum_{k=0}^{n}\binom{n-k}{2k}a_k$$

has g.f. expressible as

$$\cfrac{1}{1-x-\cfrac{\alpha_1 x^3}{1-x-\cfrac{\alpha_2 x^3}{1-x-\cdots}}}.$$

Proof. We have

$$\frac{1}{1-x}g\left(\frac{x^3}{(1-x)^2}\right) = \frac{1}{1-x}\cfrac{1}{1-\cfrac{\alpha_1\frac{x^3}{(1-x)^2}}{1-\cfrac{\alpha_2\frac{x^3}{(1-x)^2}}{1-\cdots}}}$$

$$= \cfrac{1}{1-x-\cfrac{\alpha_1\frac{x^3}{1-x}}{1-\cfrac{\alpha_2\frac{x^3}{(1-x)^2}}{1-\cdots}}}$$

$$= \cfrac{1}{1-x-\cfrac{\alpha_1 x^3}{1-x-\cfrac{\alpha_2 x^3}{1-x-\cdots}}}.$$

□

Example 26. Generalized Catalan numbers. We take $a_n = C_n$ and form the sequence

$$b_n = \sum_{k=0}^{n}\binom{n-k}{2k}c_k,$$

with g.f.

$$\frac{1}{1-x}C\left(\frac{x^3}{(1-x)^2}\right).$$

Then we can express this g.f. as

$$\cfrac{1}{1-x-\cfrac{x^3}{1-x-\cfrac{x^3}{1-x-\cfrac{x^3}{1-\cdots}}}}.$$

These are the generalized Catalan numbers A023431.

Similarly, the sequence with g.f.

$$\cfrac{1}{1-x-\cfrac{2x^3}{1-x-\cfrac{x^3}{1-x-\cfrac{x^3}{1-\cdots}}}}$$

has general term

$$\sum_{k=0}^{n} \binom{n-k}{2k}\binom{2k}{k}.$$

This is A098479.

In general, we can show that the sequence with g.f. given by

$$\cfrac{1}{1-x-\cfrac{rx^3}{1-x-\cfrac{x^3}{1-x-\cfrac{x^3}{1-\cdots}}}}$$

is the image of the power sequence r^n by the product of Riordan arrays

$$\left(\frac{1}{1-x}, \frac{x^3}{(1-x)^2}\right) \cdot (1, xC(x)).$$

Thus the g.f. of the image sequence is given by

$$\frac{1}{1-x}\frac{1}{1-r\frac{x^3}{1-x^3}C\left(\frac{x^3}{1-x^3}\right)}.$$

8 The transformation $b_n = \sum_{k=0}^{n} \binom{n+k}{3k} a_k$.

The general term of the generalized Riordan array

$$\left(\frac{1}{1-x}, \frac{x^2}{(1-x)^3}\right)$$

is given by

$$T_{n,k} = \binom{n+k}{3k}.$$

Thus if the sequence a_n has g.f. expressible in the form

$$g(x) = \cfrac{1}{1-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{1-\cfrac{\alpha_3 x}{1-\cdots}}}}$$

then the sequence

$$b_n = \sum_{k=0}^{n} \binom{n+k}{3k} a_k$$

will have g.f.

$$\frac{1}{1-x}g\left(\frac{x^2}{(1-x)^3}\right),$$

which is equal to

$$\begin{aligned}
&\frac{1}{1-x}\cfrac{1}{1-\cfrac{\alpha_1\frac{x^2}{(1-x)^3}}{1-\cfrac{\alpha_2\frac{x^2}{(1-x)^3}}{1-\cfrac{\alpha_3\frac{x^2}{(1-x)^3}}{1-\cdots}}}}\\
=\;&\cfrac{1}{1-x-\cfrac{\alpha_1\frac{x^2}{(1-x)^2}}{1-\cfrac{\alpha_2\frac{x^2}{(1-x)^3}}{1-\cfrac{\alpha_3\frac{x^2}{(1-x)^3}}{1-\cdots}}}}\\
=\;&\cfrac{1}{1-x-\cfrac{\alpha_1 x^2}{(1-x)^2-\cfrac{\alpha_2\frac{x^2}{1-x}}{1-\cfrac{\alpha_3\frac{x^2}{(1-x)^3}}{1-\cdots}}}}\\
=\;&\cfrac{1}{1-x-\cfrac{\alpha_1 x^2}{(1-x)^2-\cfrac{\alpha_2 x^2}{1-x-\cfrac{\alpha_3\frac{x^2}{(1-x)^2}}{1-\cdots}}}}.
\end{aligned}$$

Thus we arrive at the

Proposition 27. *In the circumstances above, the sequence*

$$b_n=\sum_{k=0}^{n}\binom{n+k}{3k}a_k$$

has g.f. expressible in the form

$$\cfrac{1}{1-x-\cfrac{\alpha_1 x^2}{(1-x)^2-\cfrac{\alpha_2 x^2}{1-x-\cfrac{\alpha_3 x^2}{(1-x)^2-\cdots}}}}.$$

Example 28. Number of Dyck paths of semi-length n with no $UUDD$. We let $a_n = c_n$. Then the sequence with general term

$$b_n = \sum_{k=0}^{n} \binom{n+k}{3k} c_k$$

has g.f. $\frac{1}{1-x} C\left(\frac{x^2}{(1-x)^3}\right)$ expressible as

$$\cfrac{1}{1-x-\cfrac{x^2}{(1-x)^2-\cfrac{x^2}{1-x-\cfrac{x^2}{(1-x)^2-\cdots}}}}.$$

This is A086581 (number of Dyck paths of semi-length n with no $UUDD$).

9 Bi-variate continued fractions and number triangles

We have seen that

$$\left(\frac{1}{1-rx}, \frac{x}{1-rx}\right) \cdot C(x) = \cfrac{1}{1-rx-\cfrac{x}{1-\cfrac{x}{1-rx-\cfrac{x}{1-\cfrac{x}{1-rx-\cfrac{x}{1-\cdots}}}}}}.$$

Now treating r as an independent variable (and writing it as y), we consider the bi-variate expression

$$g(x,y) = \cfrac{1}{1-xy-\cfrac{x}{1-\cfrac{x}{1-xy-\cfrac{x}{1-\cfrac{x}{1-xy-\cfrac{x}{1-\cdots}}}}}}. \tag{4}$$

This is the bi-variate generating function of the number triangle with general term

$$[k \le n]\binom{n}{k} c_{n-k},$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 5 & 6 & 3 & 1 & 0 & 0 & \dots \\ 14 & 20 & 12 & 4 & 1 & 0 & \dots \\ 42 & 70 & 50 & 20 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A124644. The image of the sequence r^n by this matrix is given by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} c_{n-k} r^k = \sum_{k=0} \binom{n}{k} c_k r^{n-k},$$

using $\binom{n}{k} = \binom{n}{n-k}$. Thus applying the matrix with bi-variate generating function given by Eq. (4) to the sequence r^n is equivalent to calculating the r-th binomial transformation of the Catalan numbers C_n.

This example may be generalized in many ways.

10 The transformation $b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} r^{n-2k} a_{n-2k}$.

Setting $y = x$ in Eq. (4) of the last section gives us the generating function of the diagonal sums of the matrix. Thus the sequence with general term

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} c_{n-2k}$$

has generating function expressible as

$$\cfrac{1}{1 - x^2 - \cfrac{x}{1 - \cfrac{x}{1 - x^2 - \cfrac{x}{1 - \cdots}}}}.$$

This is A105864. By the construction above, it is the result of applying the Riordan array

$$\left(\frac{1}{1-x^2}, \frac{x}{1-x^2} \right)$$

to the Catalan numbers. In fact, we have the following proposition :

Proposition 29. *Let a_n be a sequence with generating function expressible as*

$$\cfrac{1}{1 - \cfrac{\alpha_1 x}{1 - \cfrac{\alpha_2 x}{1 - \cdots}}}.$$

Then the sequence $b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a_{n-2k}$ *which results from applying the Riordan array*

$$\left(\frac{1}{1-rx^2}, \frac{x}{1-rx^2}\right)$$

to a_n *will have g.f. of the form*

$$\cfrac{1}{1-rx^2-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{1-rx^2-\cfrac{\alpha_3 x}{1-\cfrac{\alpha_4 x}{1-rx^2-\cfrac{\alpha_5 x}{1-\cdots}}}}}}.$$

Example 30. A transform of the large Schröder numbers. The large Schröder numbers have g.f. expressible as

$$\cfrac{1}{1-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{x}{1-\cdots}}}}}.$$

Thus the sequence with general term

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} S_{n-2k}$$

has g.f. given by

$$\cfrac{1}{1-x^2-\cfrac{2x}{1-\cfrac{x}{1-x^2-\cfrac{2x}{1-\cfrac{x}{1-x^2-\cdots}}}}}.$$

This is equal to

$$\frac{1-x-x^2-\sqrt{1-6x-x^2+6x^3+x^4}}{2x(1-x^2)}.$$

The special case $r=-1$ which corresponds to the Riordan array $\left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)$ corresponds to the so-called "Chebyshev transform". Thus the transform of c_k for this matrix has general term

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} c_{n-2k},$$

reminiscent of the formula for the Chebyshev polynomials of the second kind. This sequence is A101499. By the above, it has generating function

$$\cfrac{1}{1+x^2-\cfrac{x}{1-\cfrac{x}{1+x^2-\cfrac{x}{1-\cfrac{x}{1+x^2-\cdots}}}}}.$$

11 The Deleham construction

The Deleham construction is a means of using bi-variate continued fraction generating functions, based on two base sequences, to construct number triangles. Many triangles of classical importance may be so constructed. Numerous examples are to be found in [10]. For the purposes of this note, we can define the *Deleham construction* as follows. Given two sequences r_n and s_n,we use the notation

$$r\,\Delta\,s=[r_0,r_1,r_2,\ldots]\,\Delta\,[s_0,s_1,s_2,\ldots]$$

to denote the number triangle whose bi-variate generating function is given by

$$\cfrac{1}{1-\cfrac{(r_0x+s_0xy)}{1-\cfrac{(r_1x+s_1xy)}{1-\cfrac{(r_2x+s_2xy)}{1-\cdots}}}}.$$

We furthermore define

$$r\,\Delta^{(1)}\,s=[r_0,r_1,r_2,\ldots]\,\Delta^{(1)}\,[s_0,s_1,s_2,\ldots]$$

to denote the number triangle whose bi-variate generating function is given by

$$\cfrac{1}{1-(r_0x+s_0xy)-\cfrac{(r_1x+s_1xy)}{1-\cfrac{(r_2x+s_2xy)}{1-\cdots}}}.$$

See A084938 for the original definition.

Example 31. The Narayana triangles. Three common versions of the Narayana triangle can be expressed as follows :

$$[1,0,1,0,1,0,1,\ldots]\,\Delta\,[0,1,0,1,0,1,\ldots]$$

which is A131198,

$$[0,1,0,1,0,1,\ldots]\,\Delta\,[1,0,1,0,1,0,1,\ldots],$$

which is the reverse of that matrix, and

$$[0,1,0,1,0,1,\ldots]\,\Delta^{(1)}\,[1,0,1,0,1,0,1,\ldots]$$

which is A090181.

We have the following result:

Theorem 32. *The first column of the Deleham array*

$$[r_0,r_1,r_2,r_3,\ldots]\,\Delta\,[s_0,s_1,s_2,s_3,\ldots]$$

has g.f.

$$\cfrac{1}{1-\cfrac{r_0x}{1-\cfrac{r_1x}{1-\cfrac{r_2x}{1-\cdots}}}}.$$

The row sums of the array have g.f.

$$\cfrac{1}{1-\cfrac{(r_0+s_0)x}{1-\cfrac{(r_1+s_1)x}{1-\cfrac{(r_2+s_2)x}{1-\cdots}}}}.$$

The diagonal sums of the array have g.f.

$$\cfrac{1}{1-\cfrac{(r_0x+s_0x^2)}{1-\cfrac{(r_1x+s_1x^2)}{1-\cfrac{(r_2x+s_2x^2)}{1-\cdots}}}}.$$

The product of the array with $\mathbf{B}$ *has generating function*

$$\cfrac{1}{1-\cfrac{((r_0+s_0)x+s_0xy)}{1-\cfrac{((r_1+s_1)x+s_1xy)}{1-\cfrac{((r_2+s_2)x+s_2xy)}{1-\cdots}}}}=\cfrac{1}{1-\cfrac{r_0x+s_0x(1+y)}{1-\cfrac{r_1x+s_1x(1+y)}{1-\cfrac{r_2x+s_2x(1+y)}{1-\cdots}}}},$$

and is thus the Deleham array

$$(r+s)\,\Delta\, s.$$

The product of $\mathbf{B}$ *and the array has generating function*

$$\cfrac{1}{1-x-\cfrac{(r_0x+s_0xy)}{1-\cfrac{(r_1x+s_1xy)}{1-x-\cfrac{(r_2x+s_2xy)}{1-\cdots}}}}.$$

Proof. The g.f. of the first column is obtained by setting $y=0$ in the bivariate g.f. Similarly, the g.f. of the row sums is obtained by setting $y=1$, while that of the diagonal sums is found by setting $y=x$.

The product of the array $r\,\Delta\, s$ and $\mathbf{B}$ has g.f. given by

$$(1,x,x^2,\dots)\,(r\,\Delta\, s)\cdot\mathbf{B}\,(1,y,y^2,\dots)^\top.$$

But this is

$$(1,x,x^2,\dots)\,(r\,\Delta\, s)\,(1,1+y,(1+y)^2,\dots)^\top$$

which by assumption is

$$\cfrac{1}{1-\cfrac{r_0x+s_0x(1+y)}{1-\cfrac{r_1x+s_1x(1+y)}{1-\cfrac{r_2x+s_2x(1+y)}{1-\cdots}}}}.$$

The g.f. of the binomial transform of the array (that is, of the product of $\mathbf{B}$ and $\mathbf{r}\,\Delta\,\mathbf{s}$) will be given by

$$\frac{1}{1-x}\cfrac{1}{1-\cfrac{(r_0+s_0y)\frac{x}{1-x}}{1-\cfrac{(r_1+s_1y)\frac{x}{1-x}}{1-\cfrac{(r_2+s_2y)\frac{x}{1-x}}{1-\cdots}}}},$$

which simplifies to

$$\cfrac{1}{1-x-\cfrac{(r_0x+s_0xy)}{1-\cfrac{(r_1x+s_1xy)}{1-x-\cfrac{(r_2x+s_2xy)}{1-\cdots}}}}.$$

□

Example 33. **A088874**. The number triangle

$$[0,2,6,12,20,30,\ldots]\,\Delta\,[1,2,3,4,5,6,\ldots]$$

with g.f.

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{(2x+2xy)}{1-\cfrac{(6x+3xy)}{1-\cfrac{(10x+4xy)}{1-\cdots}}}}}$$

is studied in [4].

The Deleham construction leads to many interesting triangular arrays of numbers. The field of *associahedra* [1, 2, 3, 8] is rich in such triangles, including the Narayana triangle. We finish with some examples from this area. An associahedron is a special type of polytope. The f-vector is a vector $(f_{-1}, f_0, \ldots, f_{n-1})$ where f_i denotes the number of i-dimensional faces. The unique "(-1)-dimensional" face is the empty face. The h-vector $(h_0, h_1, \ldots, h_n)$ is determined from the f-vector by a process equivalent to that described below. In the sequel, A_n and B_n refer to standard root systems connected to rotation groups [3].

Example 34. The coefficient array for the f-vector for B_n. The triangle with general term

$$\binom{n}{k}\binom{n+k}{k}=\binom{n+k}{2k}\binom{2k}{k},$$

which is A063007, is the coefficient array for the f-vector of B_n [2]. In other words, the f-vector for B_n is given by $\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k$. By our previous results, the bi-variate generating function of this triangle is given by

$$\cfrac{1}{1-x-\cfrac{2xy}{1-x-\cfrac{xy}{1-x-\cfrac{xy}{1-x-\cfrac{xy}{1-x-\cdots}}}}}.$$

This may also be expressed as

$$\cfrac{1}{1-x-\cfrac{2xy}{1-\cfrac{x+xy}{1-\cfrac{xy}{1-\cfrac{x+xy}{1-\cdots}}}}},$$

which is

$$[1,0,1,0,1,0,\ldots]\,\Delta^{(1)}\,[0,2,1,1,1,1,\ldots].$$

Example 35. The h-vector array for B_n. Reversing the above array to get the array

$$[0,2,1,1,1,1,\ldots]\,\Delta^{(1)}\,[1,0,1,0,1,0,\ldots],$$

we see that this array, which has general term $\binom{n}{k}\binom{2n-k}{n}$, has generating function

$$\cfrac{1}{1-xy-\cfrac{2x}{1-\cfrac{x+xy}{1-\cfrac{x}{1-\cfrac{x+xy}{1-\cfrac{x}{1-\cdots}}}}}},$$

or equivalently,

$$\cfrac{1}{1-xy-\cfrac{2x}{1-xy-\cfrac{x}{1-xy-\cfrac{x}{1-xy-\cfrac{x}{1-xy-\cdots}}}}}.$$

Taking the product of this matrix with $\mathbf{B}^{-1}$ we obtain the matrix with general term $\binom{n}{k}^2$. This is the h-vector array for B_n. Its generating function is thus expressible as

$$\cfrac{1}{1-x(y-1)-\cfrac{2x}{1-\cfrac{x+x(y-1)}{1-\cfrac{x}{1-\cfrac{x+x(y-1)}{1-\cfrac{x}{1-\cdots}}}}}},$$

or

$$\cfrac{1}{1-xy+x-\cfrac{2x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}}}.$$

It may also be expressed as

$$\cfrac{1}{1-xy+x-\cfrac{2x}{1-xy+x-\cfrac{x}{1-xy+x-\cfrac{x}{1-xy+x-\cfrac{x}{1-xy+x-\cdots}}}}}.$$

This is thus the Deleham array

$$[-1,2,0,1,0,1,\ldots]\,\Delta^{(1)}\,[1,0,1,0,1,0,\ldots].$$

Example 36. The f- and h-vectors for A_n. The triangle with general term

$$\frac{1}{k+1}\binom{n}{k}\binom{n+k+2}{k}$$

is given by

$$[1,0,1,0,1,\ldots] \quad \Delta^{(1)} \quad [1,1,1,1,\ldots].$$

This is the coefficient array for the f-vector for A_n [1, 2]. We recall that

$$[1,0,1,0,1,\ldots]\,\Delta\,[1,1,1,1,\ldots]$$

has generating function

$$\cfrac{1}{1-\cfrac{x+xy}{1-\cfrac{xy}{1-\cfrac{x+xy}{1-\cdots}}}},$$

and thus the triangle

$$[1,0,1,0,1,\ldots]\,\Delta^{(1)}\,[1,1,1,1,\ldots]$$

has generating function

$$\cfrac{1}{1-(x+xy)-\cfrac{xy}{1-\cfrac{x+xy}{1-\cfrac{xy}{1-\cdots}}}}.$$

This is the array A033282 that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 5 & 5 & 0 & 0 & 0 & \ldots \\ 1 & 9 & 21 & 14 & 0 & 0 & \ldots \\ 1 & 14 & 56 & 84 & 42 & 0 & \ldots \\ 1 & 20 & 120 & 1300 & 330 & 132 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Reversing this triangle to get the triangle with general element

$$[k \le n]\frac{1}{n-k+1}\binom{n}{k}\binom{2n-k+2}{n-k},$$

and then forming the product of this matrix and $\mathbf{B}^{-1}$, we obtain the coefficient array of the h-vector for A_n [3]. This turns out to be the version of the triangle of Narayana numbers which begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & \dots \\ 1 & 15 & 50 & 50 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

and has generating function

$$\cfrac{1}{1-x-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}}.$$

Example 37. The Narayana numbers A090181. In this example, we start with the number array with general element

$$\binom{n+k}{2k}c_k = \frac{1}{k+1}\binom{n}{k}\binom{n+k}{k}.$$

This is A088617, whose terms count the number of Schröder paths from $(0,0)$ to $(2n,0)$ with k up-steps. It has generating function

$$\cfrac{1}{1-x-\cfrac{xy}{1-x-\cfrac{xy}{1-x-\cfrac{xy}{1-x-\cfrac{xy}{1-x-\cdots}}}}}.$$

This may also be expressed as

$$\cfrac{1}{1-x-\cfrac{xy}{1-\cfrac{x+xy}{1-\cfrac{xy}{1-\cfrac{x+xy}{1-\cdots}}}}},$$

which is

$$[1,0,1,0,1,0,\ldots]\,\Delta^{(1)}\,[0,1,1,1,1,1,\ldots].$$

Reversing the above array to get the array

$$[0,1,1,1,1,1,\ldots]\,\Delta^{(1)}\,[1,0,1,0,1,0,\ldots],$$

we see that this array, which has general term

$$[k\le n]\binom{2n-k}{k}c_{n-k}=[k\le n]\frac{1}{n-k+1}\binom{n}{k}\binom{2n-k}{n-k},$$

has generating function

$$\cfrac{1}{1-xy-\cfrac{x}{1-\cfrac{x+xy}{1-\cfrac{x}{1-\cfrac{x+xy}{1-\cfrac{x}{1-\cdots}}}}}},$$

or equivalently,

$$\cfrac{1}{1-xy-\cfrac{x}{1-xy-\cfrac{x}{1-xy-\cfrac{x}{1-xy-\cfrac{x}{1-xy-\cdots}}}}}.$$

Taking the product of this matrix with $\mathbf{B}^{-1}$ we obtain the matrix of the Narayana numbers that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 6 & 6 & 1 & 0 & \ldots \\ 0 & 1 & 10 & 20 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is the Narayana triangle A090181. Its generating function is thus expressible as

$$\cfrac{1}{1-x(y-1)-\cfrac{x}{1-\cfrac{x+x(y-1)}{1-\cfrac{x}{1-\cfrac{x+x(y-1)}{1-\cfrac{x}{1-\cdots}}}}}},$$

or

$$\cfrac{1}{1-xy+x-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}}}.$$

It may also be expressed as

$$\cfrac{1}{1-xy+x-\cfrac{x}{1-xy+x-\cfrac{x}{1-xy+x-\cfrac{x}{1-xy+x-\cfrac{x}{1-xy+x-\cdots}}}}}.$$

This is thus the Deleham array

$$[-1,1,0,1,0,1,\ldots]\,\Delta^{(1)}\,[1,0,1,0,1,0,\ldots].$$

In fact, it is also expressible as the Deleham array

$$[0,1,0,1,0,1,\ldots]\,\Delta\,[1,0,1,0,1,0,\ldots]$$

with generating function

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}}.$$

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