Riordan Arrays, Elliptic Functions and their Applications

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Declaration of Authorship

I, ARNAULD MESINGA MWAFISE, declare that this thesis titled, 'Riordan Arrays, Elliptic Functions and their Applications' and the work presented in it are my own. I confirm that:

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Abstract

Riordan arrays have been used as a powerful tool for solving applied algebraic and enumerative combinatorial problems from a number of different settings in pure and applied mathematics. This thesis establishes relationships between elliptic functions and Riordan arrays leading to new classes of Riordan arrays which here are called elliptic Riordan arrays. These elliptic Riordan arrays were found in many cases to be useful constructs in generating combinatorially and algebraically significant sequences based on their corresponding trigonometric and hyperbolic forms. In addition, in some cases the elliptic Riordan arrays presented interesting structural patterns that were further investigated. By exploring elliptic Riordan arrays more closely with respect to other fields, several new applications of Riordan arrays associated with physics and engineering are illustrated. Furthermore, other non-elliptic type Riordan arrays having important applications are also presented based on the connection established in the thesis between Riordan arrays and the analytic solutions to some of the families of the Sturm-Liouville differential equations.

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Notations and Abbreviations

- **g.f** Generating Function.
- **e.g.f** Exponential generating function.
- **o.g.f** Ordinary generating functions.
- **f.p.s** Formal power series.
- $\equiv$ Equivalence relation.
- $x \equiv a(mod\, b)$ Congruent relation.
- $\times$ Multiplication.
- $\mathbb{Z}$ The set of Integers.
- $\mathbb{N}$ The set of Natural numbers.
- $\mathbb{R}$ The set of Real numbers.
- $\mathbb{C}$ The set of complex numbers.
- ODE Ordinary Differential Equation.
- PDE Partial Differential Equation.
- $[z^n]f(z)$ The coefficient of z^n in the power series $f(z)$.
- $\bar{f}(z)$ or $\text{Rev}(f(z))$ the series reversion of the series $f(z)$.
- $\mathbf{R}$ A Riordan array.
- $\bar{\mathbf{R}}$ The Riordan matrix such that $\bar{\mathbf{R}}_{n,k} = R_{n+1,k}$.
- (**Axxxx**) A number. The Online Encyclopedia of Integer Sequences (OEIS)
- (g, f) An ordinary Riordan array.
- $[g, f]$ An exponential Riordan array.
- $\lfloor x \rfloor$ The Floor Function.
- m Elliptic modulus.

- $K(m)$ Complete elliptic integral of the first kind.
- $F(\phi, m)$ Elliptic integral of the first kind.
- $E(m)$ Complete elliptic integral of the second kind.
- $E(\phi, m)$ Elliptic integral of the second kind.
- $\wp$ Weierstrass elliptic function.
- ζ Zeta pseudo-elliptic function.
- σ Sigma psedo-elliptic function.
- sm Dixonian elliptic sine function.
- cm Dixonian elliptic cosine function.
- $\mathrm{sn}(z, m) \equiv \mathrm{sn}(z|m)$ Elliptic sine.
- $\mathrm{cn}(z, m) \equiv \mathrm{cn}(z|m)$ Elliptic cosine.
- $\mathrm{dn}(z, m) \equiv \mathrm{dn}(z|m)$ Difference function.
- $\vee$ Or.
- s.t such that
- $\forall$ For every.
- $\in$ An element of.
- w.r.t with respect to.

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Chapter 1

Introduction to Riordan Arrays and Elliptic Functions

1.1 Background of the study

In this chapter we shall discuss the theory of Riordan arrays and elliptic functions which are two of the main areas of research in this work. The study of Riordan arrays uses extensively the interchangeable notions of sequences, formal power series, generating functions and in some cases the Lagrange inversion theorem. These concepts are briefly described below.

- A **sequence** is a mapping from the set $\mathbb{N}_0$ of natural numbers into some other set of numbers such as the set of Real numbers $\mathbb{R}$, Natural numbers $\mathbb{N}_0$, Rational numbers $\mathbb{Q}$, Complex numbers $\mathbb{C}$. That is for a mapping f the following holds

  $$f : \mathbb{N}_0 \to \mathbb{R},\ f : \mathbb{N}_0 \to \mathbb{N}_0,\ f : \mathbb{N}_0 \to \mathbb{Q},\ f : \mathbb{N}_0 \to \mathbb{C}.$$

  In this case the sequences can be denoted as $(f_k)_{k\in\mathbb{N}_0}$. In the case of a double sequence the mapping is such that

  $$f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{Z}$$

or mapped into any other well known numeric set of numbers. There are two ways in which the mapping is denoted, which are $\{f_{n,k}|n,k \in \mathbb{N}_0\}$ or $(f_{n,k})_{n,k\in\mathbb{N}_0}$. The most obvious application of a double sequence is in the display of an infinite array of numbers usually represented in matrix form. The arrangement of the elements of the array is such that the first row can be generally seen as representing the sequence of numbers $(f_{0,0}, f_{0,1}, f_{0,2}, f_{0,3}, ...)$ with the second row given by $(f_{1,0}, f_{1,1}, f_{1,2}, f_{1,3}, ...)$ with the generalized form of the sequences from subsequent rows developed using a similar pattern.

The OEIS (Online Encyclopaedia of Integer Sequences)[100] provides a useful searchable database for researchers working in the area of enumerative combinatorics to identify useful named sequences which are the outputs of combinatorial processes or instances of combinatorial objects.

- A **formal power series** (f.p.s) f over a field of characteristic 0 in the indeterminate z is an expression of the form

$$f(z) = \sum_{k=0}^{\infty} f_k z^k = f_0 + f_1 z + f_2 z^2 + f_3 z^3 + ... + f_n z^n + ...$$

  where f_n are the coefficients of the f.p.s $\forall n \in \mathbb{N}_0$. A formal power series can also be regarded as an infinite degree polynomial. The set of formal power series is denoted by $F[[z]]$ or simply $\mathcal{F}$. The order of $f(z)$, denoted $ord(f(z))$ is the smallest index r for which $f_r \neq 0$. The set of all formal power series having order exactly r is denoted $\mathcal{F}_r$.

- The **generating function**[44] represents the closed form expression corresponding to a formal power series expansion encoding information about a sequence of numbers. In enumerative combinatorics, the nature of the counting problem will determine if the generating function will be of either the ordinary or the exponential type. The exponential type normally occurs when the order of the elements being counted is important. The generating function is a more elegant way of working with sequences of varying levels of complexities in order to uncover useful patterns and for mathematical manipulation purposes. Generating functions are sometimes called generating series as it can alternatively be viewed as a sequence of terms generating the sequence of term coefficients.

- The **Lagrange Inversion Formula** can be applied in a variety of ways

[74, 88]:

- Determining the generating function of many combinatorial sequences.
- In the extraction of the coefficients of a formal power series.
- For the computation of combinatorial sums.
- In the process of carrying out the inversion of combinatorial identities.

If $w = w(t)$ is a formal power series satisfying the relation that $w = t\phi(w)$ with $\phi(0) \neq 0$ then the Lagrange inversion formula [63] states that

$$[t^n]w(t)^k = \frac{k}{n}[t^{n-k}]\phi(t)^n.$$

Setting $f(t) = t/\phi(t) \implies f(w(t)) = w/\phi(w) = t$, therefore $w(t)$ can be considered the compositional inverse of $f(t)$. The multivariate version of the LIF is given in [37]. Some useful techniques for the extraction of coefficients in formal power series and the application of the Lagrange inversion formula have been applied to a variety of combinatorial problems [105].

This thesis document will be divided into twelve chapters. The first chapter will introduce the concept of Riordan arrays and will give a brief introduction to the notion of elliptic functions. Chapters 2, 3, 4 will give the Riordan array representation of the three main types of elliptic functions seperately. The Riordan arrays described in Chapter 2 and 3 in particular, will be subdivided into various new classes of Riordan array subgroups parameterized by the elliptic modulus m. In chapter 5 we will highlight interesting sub-matrices and new Riordan arrays that can be derived from some existing Riordan arrays based on their structural patterns. In chapter 6 the relationship between Riordan arrays and the solutions to some useful systems of differential equations that have applications in mathematical physics and engineering will be highlighted as a preview to some other interesting applications in subsequent chapters. Chapters $7, 8, 9$ will identify Riordan arrays that are constructed based on the solutions arising from the applications of elliptic functions in mathematical physics. Chapter 10 will focus on other applications of Riordan arrays which represent the solution of the quantum mechanical oscillator. In chapter 11 we will use Riordan arrays in the implementation of the Bessel filters and Elliptic filters in signal processing. The last chapter will provide the main conclusions and the future direction for further research based on the new results presented in this work. There is also an appendix section with some relevant symbolic codes using *Mathematica* software from which the various investigations were carried out in support of this work.

1.2 A brief historical time-line on the development of Riordan arrays

Riordan arrays were originally identified as a tool for solving combinatorial enumeration problems, particularly in lattice path theory [21, 73, 98]. The concept of Riordan arrays is most closely related to the Lagrange inversion theorem formulated by Lagrange [63] in 1776 and to the umbral calculus developed in 1970's by S. Roman [87]. The points highlighted below give a summary of the key time-line in the development of the Riordan array theory.

- In 1978 Rogers [91] was inspired by L. Shapiro's work which resulted in the discovery of the triangle of numbers called the Catalan triangle generated by its first column. The Catalan triangle was the second type of array to be generated in a similar manner to the Pascal triangle which was previously the only known array with combinatorial properties. This led to the formulation of a new method in determining the generalization of a family of triangular arrays called the Renewal arrays with arithmetic properties analogous to the Catalan and the Pascal triangle. The result was the discovery of the A-sequence represented by a recursive formula (see [91]) that partially determined the entries of the new array known as the renewal array.

- A follow-up to the work of Rogers on renewal arrays was the introduction in 1991 of the Riordan Group [98] led by Louis Shapiro in collaboration with his colleagues Getu, Woan and Woodson from Howard Unversity, USA. It was named in honour of John Riordan whose pioneering research work on combinatorics culminated in the publication of his well known book titled "Combinatorial Identities"[89] which is a useful resource on problems involving the inversion of combinatorial sums. Shapiro[98] gives the initial definition of the properties of the Riordan group, its foremost theorem which is considered the fundamental theorem of Riordan arrays with the preliminary applications in evaluating combinatorial sums. Furthermore, the new theory of Riordan arrays focused on the columns and in particular algebraically describing them in terms of their associated generating functions rather than working with the umbral calculus.

- Sprugnoli [102, 103, 27, 104] expanded the applications of Riordan arrays to the proofs of combinatorial identities, evaluation and inversion of combinatorial sums.

- The introduction of the Z sequence characterization of Riordan arrays by Merlini et al. (1997) in [73] which completely solved the problem of recursively determining all the elements of a Riordan array.

- The connection between production matrices and Riordan arrays by Emeric Deutsch [31] in 2007 introduced the r and c sequence formulas analogous to the A and Z sequences to determine the recurrence coefficients and generating functions for the case of exponential Riordan arrays.

- Previous research has found broader application of Riordan arrays to the following areas:

  - Wireless communications for MIMO calculations [50]
  - Computer science for algorithm analysis [75, 6]
  - Molecular biology for RNA secondary structure enumeration [80]
  - Error correcting codes in computer science [8]
  - Chemistry [24]
  - Queueing theory relating to birth-and-death processes [22].

Over the years there has been very active research with a multitude of publications in the area of Riordan arrays. An evidence of this active research effort can be seen in the over 16300 results when the keyword "Riordan arrays" is entered into the Google search engine. Sprugnoli in 2008 [106] put together a bibliography collection of over 60 published research papers in the area. This number has however increased significantly as can be seen from current updates available in online academic search engines showing that more research papers have been published on the subject. Key researchers who have contributed enormously to the domain of Riordan arrays are: L. Shapiro [98], R. Sprugnoli [107], D. Merlini [77], P. Barry [14], A. Luzón [70], M. Morón [79], T-X He [49], G.S Cheon [23] and so forth. Their research contributions can be found on their individual profiles available on academic web sites such as ResearchGate and Google Scholar from where their featured publications on Riordan arrays can be retrieved.

1.3 Definitions and Examples of Riordan Arrays

This section will provide the definitions to the three types of Riordan arrays together with some of their examples.

1.3.1 Definitions of Riordan arrays

The original forms of Riordan arrays comprise of the Ordinary and the Exponential Riordan array and these two types are the most often used. A third type of Riordan array known as the Generalized Riordan array was introduced later.

Definition : An **Ordinary Riordan array** can be described as an infinite lower triangular array $(d_{n,k})_{0\leq k\leq n}$ defined by a pair of formal power series represented by their analytic generating functions $d(t) = \sum_{n=0}^{\infty} d_n t^n$ and $h(t) = \sum_{n=0}^{\infty} h_n t^n$ such that the entries of the array columns are evaluated by the formula

$$d_{n,k} = [t^n]d(t)h(t)^k$$

where k is the column number. It is denoted $(d(t), h(t))$.

If all the conditions $d(0) \neq 0$, $h(0) = 0$ and $h'(0) \neq 0$ are satisfied together then it is considered a proper Riordan array which shows that it is invertible.

In the case of the exponential Riordan arrays the definition is a different variation of the ordinary Riordan arrays given in terms of ordinary generating functions which are substituted in terms of exponential generating function.

Definition An **Exponential Riordan array** is an infinite lower triangular array consisting of a pair of formal power series represented by their exponential generating functions given by

$$d(t) = 1 + \sum_{n=1}^{\infty} d_n(t^n/n!) \qquad h(t) = \sum_{n=1}^{\infty} h_n(t^n/n!)$$

with the generic element associated with the coefficients of column k evaluated by

$$d_{n,k} = \frac{n!}{k!}[t^n]d(t)h(t)^k.$$

It is denoted $[d(t), h(t)]$.

From the above definition it is considered a proper exponential Riordan array otherwise it is considered a non-proper exponential Riordan array if $h(0) \neq 0$. The importance of proper Riordan arrays will later be seen as a key condition in the sequence characterization of Riordan arrays which will be discussed in subsequent sections of this chapter. Most of the applications of this thesis will be based on exponential Riordan arrays.

Alternatively, in terms of bivariate generating function an ordinary Riordan arrays [102] is given by

$$d(t,x) = \sum_{k=0}^{\infty} d(t)h(t)^k x^k = \frac{d(t)}{1 - xh(t)}. \tag{1.1}$$

On the other hand the bivariate generating function of exponential Riordan arrays is given by

$$d_e(t,x) = \frac{1}{k!}\sum_{k=0}^{\infty} d(t)h(t)^k x^k = d(t)e^{xh(t)}. \tag{1.2}$$

The case for $x = 1$ in (1.1) and (1.2) results in the explicit formula for the row sum of a Riordan array.

Definition The definition of Riordan arrays associated with Laurent series is given in [47] as a Riordan matrix having entries given by

$$d_{n,k} = [t^n]\frac{1}{d(t^{-1})}\left(\frac{1}{h(t^{-1})}\right)^k$$

and denoted by $(1/d(t^{-1}), 1/h(t^{-1}))$.

The most recently introduced type of Riordan array is the Generalized Riordan array which is a development of the ordinary and exponential Riordan arrays.

Definition A **Generalized Riordan array** [115] consists of a Riordan array given by the pair $(d(t), h(t))$ expressed in relation to the non-zero sequence of elements c_n with initial conditions $c_0 = 1$ where

$$d(t) = \sum_{k=0}^{\infty} d_k t^k / c_k \qquad h(t) = \sum_{k=0}^{\infty} h_k t^k / c_k$$

such that its generic element from which its columns are generated can be expressed as

$$d_{n,k} = \left[\frac{t^n}{c_k}\right] d(t)h(t)^k.$$

A special feature arising from the definition of generalized Riordan arrays is that it is a more flexible concept to work with as it can be converted to the ordinary Riordan arrays if $c_n = 1$ and to exponential Riordan arrays if $c_n = n!$.

1.3.2 Some Basic Examples of Riordan arrays

All Riordan arrays are a generalization of the Pascal triangle that is equivalent to the binomial matrix $\binom{n}{k}$ which has all its leading diagonal elements equal to 1. The most basic and important example of a Riordan array which gives a good entry point into investigating the various theories, methods and techniques arising from the Riordan array concept is the Pascal triangle. The Pascal triangle consists of binomial coefficients which can be described in terms of its ordinary generating functions defined by

$$\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$$

with its general element given by $\binom{n}{k}$. The generalized form of the Pascal triangle is given by

$$\left(\frac{1}{1-mt}, \frac{t}{1-mt}\right) = \left(\frac{1}{1-t}, \frac{t}{1-t}\right)^m, m \geq 0$$

having general term $\binom{n}{k} m^{n-k}$. Its equivalent exponential Riordan array is defined by $[e^t, t]$. The exponential type of Pascal triangle can be generalized as

$$([e^t, t])^m = [e^{mt}, t]$$

such that $m = 1$ encodes the binomial matrix entries. We can apply the column definition of a Riordan array and the properties of binomial coefficients [44, 105] to determine the general formula $a_{n,k}$ of the binomial matrix as illustrated below.

$$\begin{aligned}
a_{n,k} &= [t^n]\frac{1}{1-t}\left(\frac{t}{1-t}\right)^k = \left[t^{n-k}\right]\frac{1}{(1-t)^{k+1}} \\
&= \left[t^{n-k}\right]\sum_{j=0}^{\infty}\binom{-(k+1)}{j}(-1)^j t^j \\
&= \left[t^{n-k}\right]\sum_{j=0}^{\infty}\binom{k+1+j-1}{j}t^j \\
&= \left[t^{n-k}\right]\sum_{j=0}^{\infty}\binom{k+j}{j}t^j \\
&= \binom{k+n-k}{n-k} \\
&= \binom{n}{n-k} = \binom{n}{k}.
\end{aligned}$$

This shows that a Riordan array is a generalization of the Pascal triangle.

Other well known examples of Riordan arrays that have received much attention are the Catalan arrays [91, 76] and the Fibonacci arrays [40]. The Catalan matrix is $\mathrm{C} = \left(\frac{1-\sqrt{1-4t}}{2t}, \frac{1-\sqrt{1-4t}}{2}\right)$ which has general term

$$c_{n,k} = \frac{k+1}{n+1}\binom{2n-k}{n-k}$$

forming the Riordan matrix

$$\mathrm{C} = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \end{pmatrix}.$$

An example of a generalised Riordan array is $\left((1+t^2)^{-\lambda_0}, \frac{-2t}{1+t^2}\right)$ with respect to the sequence $c_n = \frac{1}{\binom{-\lambda}{n}}$ and $\left(\frac{1-t^2}{1+t^2}, \frac{-2t}{1+t^2}\right)$. Several other examples of the generalised Riordan arrays are listed in [115].

1.4 The Group Structure of Riordan Matrices

The main operation required to determine that a set of Riordan matrices forms a group is multiplication ($*$). The multiplication rule for two Riordan arrays (g, f) and (h, l) is defined as

$$(g, f) * (h, l) = (g * (h \circ f), l \circ f)\,. \tag{1.3}$$

In Shapiro [98] the key criteria for $(R, *)$ to be considered a group under the multiplication operation is stated based on the following conditions listed below.

1. The existence of $(1, t)$ as the identity element.

2. The inverse element is given by

$$(d(t), h(t))^{-1} = \left(\frac{1}{d(\bar{h}(t))}, \bar{h}(t)\right). \tag{1.4}$$

   We note that for a power series $h(t) = \sum_{n=0}^{\infty} h_n t^n$ with $h(0) = 0$, we define the reversion or the compositional inverse of h to be the power series $\bar{h}(t)$ such that $h(\bar{h}(t)) = \bar{h}(h(t)) = t$. It can sometimes be denoted simply as $\bar{h}$ or *Rev* h.

3. The set is closed under the multiplication of matrices.

4. It is associative since multiplication of matrices is associative.

The next example shows the process of Riordan array multiplication and inverse computation.

Example Consider the Pascal triangle given by $\mathrm{P} = \left(\frac{1}{1-t}, \frac{t}{1-t}\right)$

$$\begin{aligned} \mathrm{P} * \mathrm{P} &= \left(\frac{1}{1-t}, \frac{t}{1-t}\right) * \left(\frac{1}{1-t}, \frac{t}{1-t}\right) \\ &= \left(\frac{1}{1-t}\frac{1-t}{1-2t}, \frac{t}{1-t}\frac{1-t}{1-2t}\right) \\ &= \left(\frac{1}{1-2t}, \frac{t}{1-2t}\right). \end{aligned}$$

The compositional inverse of $\frac{1}{1-t}$ in P is $\frac{t}{t+1}$. Therefore we have,

$$\begin{aligned} \mathrm{P}^{-1} &= \left(\left(\frac{1}{1-\frac{t}{1+t}} \right)^{-1}, \frac{t}{1+t} \right) \\ &= \left(\frac{1}{1+t}, \frac{t}{1+t} \right). \end{aligned}$$

The existence of the inverse element of Riordan arrays therefore makes them suitable to perform operations involving combinatorial sum inversion.

1.4.1 The Sub-groups of the Riordan Group

Some important sub-groups of the Riordan group have been presented in [99]. The various forms of the subgroups arising from the Riordan array $(d(t), h(t))$ are listed as follows:

- The **Appell** subgroup has the form $(d(t), t)$.

- The **Lagrange** subgroup has the form $(1, h(t))$. The Lagrange subgroup is also known as the Associated subgroup.

- The **Renewal** subgroup which is also referrred to as the **Bell** subgroup has the form $(d(t), td(t))$.

- The **Hitting time** subgroup was introduced by Peart and Woan in [84] and has the form $\left(\frac{th'(t)}{h(t)}, h(t)\right)$ where $h'(t)$ denotes the first derivative of $h(t)$.

- The **Checkerboard** subgroup of the Riordan array $(d(t), h(t))$ represents a Riordan array such that $d(t)$ is an even generating function and $h(t)$ is an odd generating function.

- The **Derivative** subgroup has the form $(h'(t), h(t))$.

The product of an Appell and a Lagrange Riordan array is such that

$$(d(t), t) * (1, h(t)) = (d(t), h(t)).$$

1.5 Sequence Characterization of Riordan arrays

Three key theorems have underpinned much of the study within the concept of Riordan arrays. They are the Fundamental Theorem of Riordan arrays together with the theorems on the A and Z sequences which characterizes the formation of a Riordan matrix. The A and Z sequence characterization of a Riordan array $(d(t), h(t))$ involves determining both $d(t)$ and $h(t)$ respectively that define any such array. The theorems arising from the sequence characterization of Riordan arrays present an alternative definition of a Riordan array in terms of the recursive formation of its elements. These theorems are summarized below.

1.5.1 The Fundamental Theorem of Riordan Arrays

The Fundamental Theorem of Riordan arrays is an important gateway in proving many combinatorial identities and in solving problems related to combinatorial sums. The Fundamental Theorem of Riordan arrays was put forward in the initial paper which introduced the concept of the Riordan group [98]. The Fundamental Theorem of Riordan Arrays (FTRA) is formulated as follows:

Theorem 1.5.1 *[29] Suppose $(d(t), h(t))$ is a Riordan array. Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ with A and B representing column vectors such that $A = (a_0, a_1, a_2, ...)^T$ and $B = (b_0, b_1, b_2, ..)^T$. Then $(d, h)\, A = B$, if and only if $B(t) = d(t)A(h(t))$.*

1.5.2 The A-Sequence

The A-sequence introduced by Rogers (1978) [91] characterizes the column elements after the first column.

Theorem 1.5.2 *An infinite lower triangular array $D = (d_{n,k})_{n,k\in\mathbb{N}_0}$ is a Riordan array if and only if a sequence $A = \{a_n\}_{n\in\mathbb{N}_0}$ exists such that for every $n.k \in \mathbb{N}_0$ it is true:*

$$d_{n+1,k+1} = \sum_{i=0}^{n-k} a_i d_{n,k+i}.$$

Even more, if $D = (d(t), h(t))$, then the generating function of the A sequence is such that $\mathrm{A}(t) = \sum_{i=0}^{\infty} a_i t^i$ satisfies the equation

$$h(t) = t\mathrm{A}(h(t)) \quad \Rightarrow \quad \mathrm{A}(t) = \frac{t}{\bar{h}(t)} \ .$$

1.5.3 The Z-Sequence

The Z−sequence was introduced by Merlini et al. and defined in the paper [73] as a follow-up study to completing the sequence characterization of Riordan arrays which started with the formulation of the A-sequence by Rogers in [91]. The Z-sequence characterises the elements of the first column of a proper Riordan arrays as follows:

Theorem 1.5.3 *Let $(d(t), h(t)) = (d_{n,k})_{n.k\geq 0}$. Then a unique sequence $Z = (z_0, z_1, z_2, z_3, ...)$ can be determined such that every element in column 0 excluding the element in the first row can be expressed as a linear combination of all the elements in the preceding row with the coefficients identified as the elements of the sequence Z satisfying the relation*

$$d_{n+1,0} = \sum_{i=0}^{n} z_i d_{n,i} \quad (n \in \mathbb{N}_0).$$

.

The generating function of the Z sequence satisfies the equation

$$d(t) = \frac{d_0}{1 - t(Z(h(t))} \implies \mathrm{Z}(t) = \frac{1}{\bar{h}(t)} \left(1 - \frac{d_0}{d(\bar{h}(t))}\right).$$

1.6 Production Matrices and Matrix Characterisation of Riordan Arrays

This section presents an overview of the concepts of production matrices and the matrix characterization of Riordan arrays which will often be applied in subsequent chapters. Production matrices were first introduced in the paper [30]. Production matrices are simply described as infinite matrices that can be deduced from succession rules and which allows algebraic operations to be performed [117]. The explicit formula for computing the production matrix P associated to an infinite lower triangular matrix $\mathrm{D} = (d_{n,k})$ is given by

$$\mathrm{P} = \mathrm{D}^{-1} \cdot \bar{\mathrm{D}} \tag{1.5}$$

where $\bar{\mathrm{D}}$ represents the matrix corresponding to the top row of D deleted so that $\bar{\mathrm{D}} = \mathrm{D}_{n+1,k}$. The matrix characterization of Riordan arrays which results from the production matrix P provides an alternative view of the sequence characterization of Riordan arrays discussed in section (1.5).

1.6.1 Production Matrices of Ordinary Riordan arrays

The production matrix P corresponding to an ordinary Riordan has the structure:

$$\mathrm{P} = \begin{pmatrix} z_0 & a_0 & 0 & 0 & 0 & 0 & \dots \\ z_1 & a_1 & a_0 & 0 & 0 & 0 & \dots \\ z_2 & a_2 & a_1 & a_0 & 0 & 0 & \dots \\ z_3 & a_3 & a_2 & a_1 & a_0 & 0 & \dots \\ z_4 & a_4 & a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (\mathrm{Z}(t), \mathrm{A}(t), t\mathrm{A}(t), t^2\mathrm{A}(t), t^3\mathrm{A}(t), ...)$$

where $(z_0, z_1, z_2, z_3, ...)$ and $(a_0, a_1, a_2, a_3, ...)$ are the Z and A sequences corresponding to the generating functions $\mathrm{Z}(t)$ and $\mathrm{A}(t)$ respectively. The algebraic structures of some subgroups of Riordan arrays are determined using their P−matrix characterization [48].

1.6.2 Production Matrices of Exponential Riordan arrays

The relationship between production matrices and Riordan arrays with the formulation of the r and c sequence characterisation of exponential Riordan arrays was established by Deutsch [31]. These results are described in the following statements from [10] on the production matrices of exponential Riordan arrays.

Proposition 1.6.1 *Let $A = (a_{n,k})_{n,k\geq 0} = [g(x), f(x)]$ be an exponential Riordan array and let*

$$c(y) = c_0 + c_1 y + c_2 y^2 + \dots, \qquad r(y) = r_0 + r_1 y + r_2 y^2 + \dots \tag{1.6}$$

be two formal power series such that

$$r(f(x)) = f'(x) \tag{1.7}$$

$$c(f(x)) = \frac{g'(x)}{g(x)}. \tag{1.8}$$

Then

$$(i) \qquad a_{n+1,0} = \sum_i i! c_i a_{n,i} \tag{1.9}$$

$$(ii) \qquad a_{n+1,k} = r_0 a_{n,k-1} + \frac{1}{k!} \sum_{i \geq k} i!(c_{i-k} + k r_{i-k+1}) a_{n,i} \tag{1.10}$$

or, defining $c_{-1} = 0$,

$$a_{n+1,k} = \frac{1}{k!} \sum_{i \geq k-1} i!(c_{i-k} + k r_{i-k+1}) a_{n,i}. \tag{1.11}$$

Conversely, starting from the sequences defined by (1.6), the infinite array $(a_{n,k})_{n,k \geq 0}$ *defined by (1.11) is an exponential Riordan array.*

A consequence of this proposition is that P $= (p_{i,j})_{i,j \geq 0}$ where

$$p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{i-j+1}) \qquad (c_{-1} = 0)$$

$$P = \begin{pmatrix} c_0 & r_0 & 0 & 0 & 0 & 0 & \dots \\ 1!c_1 & \frac{1!}{1!}(c_0 + r_1) & r_0 & 0 & 0 & 0 & \dots \\ 2!c_2 & \frac{2!}{1!}(c_1 + r_2) & \frac{2!}{2!}(c_0 + 2r_1) & r_0 & 0 & 0 & \dots \\ 3!c_3 & \frac{3!}{1!}(c_2 + r_3) & \frac{3!}{2!}(c_1 + 2r_2) & \frac{3!}{3!}(c_0 + 3r_1) & r_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Furthermore, the bivariate exponential generating function

$$\phi_P(t, z) = \sum_{n,k} p_{n,k} t^k \frac{z^n}{n!}$$

of the matrix P is given by

$$\phi_P(t, z) = e^{tz}(c(z) + t r(z)).$$

Note in particular that we have

$$\mathrm{r}(z) = f'(\bar{f}(z))$$

and

$$\mathrm{c}(z) = \frac{g'(\bar{f}(z))}{g(\bar{f}(z))}.$$

1.7 Orthogonal polynomials and Riordan arrays

The modern theory of orthogonal polynomials is based on the research work by Szegö [110]. The importance of orthogonal polynomials (OPs) are most often associated to the solutions of mathematical and physical problems. These polynomials have been identified in areas of study such as wave mechanics, heat conduction, electromagnetic theory, quantum mechanics, electronic filter design and mathematical statistics [83].

Definition An **orthogonal polynomial sequence**(OPS) $(p_n(x))_{n\geq 0}$ refers to a sequence of polynomials $p_n(x), n \geq 0$ of degree n, with real coefficients such that any pair in the sequence are mutually and continuously orthogonal on an interval $[x_0, x_1]$ with respect to a weight function $\omega : [x_0, x_1] \to \mathbb{R}$:

$$\int_{x_0}^{x_1} p_n(x)p_m(x)\omega(x)dx = \delta_{nm}\sqrt{h_n h_m}$$

where

$$\delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

and

$$\int_{x_0}^{x_1} p_n^2(x)\omega(x)dx = h_n.$$

We assume that ω is strictly positive and continuous on the interval (x_0, x_1). Additionally, the weight function ω satisfies the first order differential equations with polynomial coefficients. Some examples of classical orthogonal polynomials are the Hermite, Laguerre, Legendre, Jacobi, Bessel, Chebyshev [35]. Some of these polynomials will be treated in subsequent chapters. These systems of orthogonal polynomials have the following common properties below [81].

1. The weight function $\omega(x)$ on the interval of orthogonality (a, b) satisfies

the Pearson differential equation

$$\frac{\omega'(x)}{\omega(x)} = \frac{p_0 + p_1 x}{q_0 + q_1 x + q_2 x^2} \equiv \frac{A(x)}{B(x)}, \qquad x \in (a, b)$$

where the following conditions hold at the end points of the interval of orthogonality:

$$\lim_{x \to \alpha+0} \omega(x) B(x) = \lim_{x \to b-0} \omega(x) B(x) = 0.$$

2. The polynomial $y = P_n(x)$ of order n satisfies the differential equation

$$B(x) y'' + [A(x) + B'(x)] y' - n[p_1 + (n+1) q_2] y = 0.$$

3. The Rodrigues formula holds:

$$P_n(x) = \frac{c_n}{\omega(x)} \frac{d^n}{dx^n} [\omega(x) B^n(x)].$$

A key characteristic of sequences from a family of orthogonal polynomial sequences is that they satisfy a so-called 'three term recurrence' [83].

Theorem 1.7.1 *[9] A sequence of orthogonal polynomials* $\{p_n(x)\}_{n=0}^{\infty}$ *satisfies*

$$p_{n+1}(x) = (\gamma_n x - \alpha_n) p_n(x) - \beta_n p_{n-1}(x)$$

for coefficients γ_n, α_n *and* β_n *that depend on* n *but not* x*.We note that if*

$$p_1(x) = k_j x^j + k_j' x^{j-1} + \ldots \; j = 0, 1, \ldots$$

then

$$\gamma_n = \frac{k_{n+1}}{k_n}, \qquad \alpha_n = \gamma_n \left(\frac{k_{n+1}'}{k_{n+1}} - \frac{k_n'}{k_n} \right), \qquad \beta_n = \gamma_n \left(\frac{k_{n-1} h_n}{k_n h_{n-1}} \right).$$

Since the degree of $p_n(x)$ is n, the coefficient array of the polynomials is a lower triangular (infinite) matrix. In the case of monic orthogonal polynomials the leading coefficient of the polynomial $k_n = 1$ for all n.
The *moments* associated to the orthogonal polynomial sequence are the numbers

$$\mu_n = \int_{x_0}^{x_1} x^n w(x) dx.$$

$p_n(x)$, α_n ,β_n and $w(x)$ can be determined from known moments.
If Δ_n denotes the Hankel determinant $|\mu_{i+j}|_{i,j\geq 0}^n$ and $\Delta_{n,x}$ denotes the same determinant, but with the last row equal to $1, x, x^2, \ldots$ then

$$p_n(x) = \frac{\Delta_{n,x}}{\Delta_{n-1}}. \tag{1.12}$$

More generally, let $H\begin{pmatrix} u_1 & \ldots & u_k \\ v_1 & \ldots & v_k \end{pmatrix}$ be the determinant of Hankel type with (i,j)-th term $\mu_{u_i+v_j}$. Let

$$\Delta_n = H\begin{pmatrix} 0 & 1 & \ldots & n \\ 0 & 1 & \ldots & n \end{pmatrix}, \qquad \Delta_n' = H_n\begin{pmatrix} 0 & 1 & \ldots & n-1 & n \\ 0 & 1 & \ldots & n-1 & n+1 \end{pmatrix}.$$

Thus,

$$\alpha_n = \frac{\Delta_n'}{\Delta_n} - \frac{\Delta_{n-1}'}{\Delta_{n-1}}, \qquad \beta_n = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}.$$

In summary for a family of polynomials $\{p_n(x)\}_{n\geq 0}$ to be *formally orthogonal*, then there exists a linear functional $\mathcal{L}$ on polynomials satisfying the following conditions:

1. $p_n(x)$ is a polynomial of degree n,
2. $\mathcal{L}(p_n(x)p_m(x)) = 0$ for $m \neq n$,
3. $\mathcal{L}(p_n^2(x)) \neq 0$.
4. The sequence of numbers $\mu_n = \mathcal{L}(x^n)$ is called the sequence of moments of the family of orthogonal polynomials defined by $\mathcal{L}$.
5. There exists an OPS w.r.t $\mathcal{L}$ if and only if

   $$\Delta_n \neq 0,\ \forall n \in \mathbb{N}_0.$$

   In particular, for the given moment sequence $(\mu_n)_{n=0}^{\infty}$

   $$\Delta_n = \det[\mu_{j+k}]_{j,k=0}^n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_n \\ \mu_1 & \mu_2 & \ldots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \ldots & \mu_{2n} \end{vmatrix}.$$

The following well-known results listed below specify the links between orthogonal polynomials, three term recurrences, the recurrence coefficients and the g.f. of the moment sequence of the orthogonal polynomials.

Theorem 1.7.2 *[114, 72], Theorem* 50.1*)(Favard's Theorem). Let* $(p_n(x))_{n\geq 0}$ *be a sequence of monic polynomials, the polynomial* $p_n(x)$ *having degree* $n = 0, 1, \ldots$ *Then the sequence* $(p_n(x))$ *is (formally) orthogonal if and only if there exist sequences* $(\alpha_n)_{n\geq 0}$ *and* $(\beta_n)_{n\geq 1}$ *with* $\beta_n \neq 0$ *for all* $n \geq 1$*, such that the three-term recurrence*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \textit{for} \quad n \geq 1,$$

holds, with initial conditions $p_0(x) = 1$ *and* $p_1(x) = x - \alpha_0$*.*

Theorem 1.7.3 *[114], Theorem* 51.1*). Let* $(p_n(x))_{n\geq 0}$ *be a sequence of monic polynomials, which is orthogonal with respect to some functional* $\mathcal{L}$*. Let*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \textit{for} \quad n \geq 1,$$

be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the generating function

$$g(x) = \sum_{k=0}^{\infty} \mu_k x^k$$

for the moments $\mu_k = \mathcal{L}(x^k)$ *satisfies*

$$g(x) = \cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}.$$

Given a family of monic orthogonal polynomials

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0,$$

we can write

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k.$$

Then we have

$$\sum_{k=0}^{n+1} a_{n+1,k}x^k = (x-\alpha_n)\sum_{k=0}^{n} a_{n,k}x^k - \beta_n \sum_{k=0}^{n-1} a_{n-1,k}x^k$$

from which we deduce

$$a_{n+1,0} = -\alpha_n a_{n,0} - \beta_n a_{n-1,0} \tag{1.13}$$

and

$$a_{n+1,k} = a_{n,k-1} - \alpha_n a_{n,k} - \beta_n a_{n-1,k} \tag{1.14}$$

We note that if α_n and β_n are constant, equal to α and β, respectively, then the sequence $(1, -\alpha, -\beta, 0, 0, \ldots)$ forms an A-sequence for the coefficient array.

Proposition 1.7.4 *[85] If $L = (g(x), f(x))$ is a Riordan array and $P = S_L$ is tridiagonal, then necessarily*

$$P = S_L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ b_1 & a & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & a & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & a & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & a & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & a & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$f(x) = Rev\frac{x}{1+ax+bx^2} \qquad \text{and} \qquad g(x) = \frac{1}{1-a_1x-b_1xf},$$

and vice-versa.

This leads to the important corollary

Corollary 1.7.5 *[11] If $L = (g(x), f(x))$ is a Riordan array and $P = S_L$ is* *tridiagonal, with*

$$P = S_L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ b_1 & a & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

then L^{-1} is the coefficient array of the family of orthogonal polynomials $p_n(x)$ where $p_0(x) = 1$, $p_1(x) = x - a_1$, and

$$p_{n+1}(x) = (x - a_n)p_n(x) - b_n p_{n-1}(x), \qquad n \geq 1,$$

where b_n is the sequence $, b_1, b, b, b, \dots$.

Theorem 1.7.6 *[11] A Riordan array $L = (g(x), f(x))$ is the inverse of the coefficient array of a family of orthogonal polynomials if and only if its production matrix $P = S_L$ is tri-diagonal.*

Proposition 1.7.7 *Let $L = (g(x), f(x))$ be a Riordan array with tri-diagonal production matrix S_L. Then*

$$[x^n]g(x) = \mathcal{L}(x^n),$$

where $\mathcal{L}$ is the linear functional that defines the associated family of orthogonal polynomials.

Based on (1.7.7) the moment sequence is given by the first column of the Riordan array L whenever the conditions established for the orthogonality of a Riordan array is satisfied.

1.8 Introduction to the theory of Elliptic Functions

In the mathematical area of complex analysis, an **elliptic function** is considered to be a function of a complex variable which is meromorphic on an open set and is doubly periodic [118, 58]. On the other hand, an elliptic integral from which the development of elliptic functions originates, represents an integral of the form

$$\int R(x, \sqrt{p(x)})dx$$

where $R(x, w)$ is a rational function in two variables and $p(x)$ is a polynomial of degree 3 or 4 having no repeated roots.

The development of the theory of elliptic functions is historically linked to earlier theoretical studies of elliptic integrals beginning from the 17^{th} century as outlined in [39]. During this period many interesting integrals arose during the process of solving mechanical problems. One of the most notable of such problems was when Wallis in 1655 began studying the arc length of an ellipse by considering the lengths of various cycloids and then relating them to the arc length of an ellipse. This culminated in the publication of an infinite series expansion of the arc length of an ellipse. This result was followed by the work of the mathematician Jacob Bernoulli ($1654 - 1705$) who in 1694 extended the results on the theory of elliptic integrals. The result of J. Bernoulli derived from his experiment on compressing an elastic rod at its ends, showed that the resulting curve satisfied the equation [39]

$$\frac{ds}{dt} = \frac{1}{\sqrt{1 - t^4}}.$$

This was followed by the introduction of the lemniscate curve given by

$$(x^2 + y^2)^2 = (x^2 - y^2).$$

The arc length of the lemniscate curve was subsequently determined by computing an elliptic integral referred to as the lemniscate integral given by

$$\int_0^x \frac{dt}{\sqrt{1 - t^4}}.$$

The above integral can be expressed in terms of arcsine x given by

$$\sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1 - t^2}}.$$

Another important observation by Jacob Bernoulli in 1694 found that the integral

$$\int \frac{t^2 dt}{\sqrt{1-t^4}}$$

could not be expressed in terms of existing transcendental functions. This nature of problem served later as one of the motivations for further research on such integrals.

The origins of the main theory of elliptic functions can be traced back to the 19th century work on integral calculus by the famous mathematician Niels H. Abel (1802 − 1829) [51]. His most remarkable achievement in this area was implementing the main technique of inverting elliptic integrals which led to elliptic functions. Abel using the main results of Euler, Lagrange and Legendre on elliptic integrals defines his elliptic function $\varphi\alpha = x$ published in his memoir titled *Recherches Sur Les Fonctions Elliptiques*(1828)[1] by the relation

$$\alpha = \int_o^x \frac{dt}{\sqrt{(1-c^2t^2)(1+e^2t^2)}} \tag{1.15}$$

where c and e are real numbers. In addition, the definition (1.15) is equivalent to the differential equation

$$\varphi'\alpha = \sqrt{(1-c^2\varphi^2\alpha)(1+e^2\varphi^2\alpha)}.$$

Abel extends the definition of $\varphi\alpha = x$ to the entire complex domain using the addition theorem based on Euler addition theorem for elliptic integrals.
The main motivation to investigate the inverse of elliptic integrals was due to the three forms of elliptic integrals put forward by Legendre in his publication *Traité des fonctions elliptiques et des intégrales eulériennes* (Paris,1825)[67]. These forms are

1. $F(\phi,k) = \int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \equiv \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ (if $x = \sin\phi$)

2. $E(\phi,k) = \int_0^\phi \sqrt{1-k^2\sin^2\phi}\,d\phi \equiv \int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}dx$ (if $x = \sin\phi$)

3. $\Pi(\phi,n,k) = \int_0^\phi \frac{d\phi}{(1+n\sin^2\phi)\sqrt{1-k^2\sin^2\phi}}$ which is equivalent to $\int_0^x \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$ (if $x = \sin\phi$).

These integrals are called elliptic integrals of the first, second and third kind respectively. These integrals were the outcome of the effort by Legendre to compute the arc length of an ellipse. N.H. Abel (1802 − 1829) contributed by inverting the Legendre elliptic integral of the first kind. This led to the introduction of the possibility of working with complex variables rather than the restriction to only real variables. The inverse function derived from inverting the elliptic integral is considered the simplest elliptic function. During the same period when Abel put forward his work on the inversion of elliptic integrals another mathematician Carl G. Jacobi(1804 − 1851) also worked in the same area. In 1829 Jacobi introduced the **Jacobi elliptic functions** denoted as $\text{sn } u, \text{cn } u, \text{dn } u$. A key characteristic of these functions is that they satisfy the equation describing quartic elliptic curves given by

$$(y')^2 = (1 - x^2)(1 - k^2x^2)$$

Jacobi's work on elliptic functions primarily focused on the inverse function as presented in the sequence below:

$$\phi = \text{am } u \qquad (\text{am } u \text{ is the Jacobi amplitude function})$$

of the integral of the first type

$$u = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

This leads to the 3 single valued functions derived from the inverse function *am u* which is multivalued, these functions are

$$\sin \phi = \sin \ \text{am } u$$

$$\cos \phi = \cos \ \text{am } u$$

$$\triangle \phi = \triangle \ \text{am } u.$$

The standard notation of Jacobi elliptic functions are denoted from the results above as follows

$$\sin \ \text{am } u \equiv \text{sn } u$$

$$\cos \ \text{am } u \equiv \text{cn } u$$

$$\triangle \ \text{am } u \equiv \text{dn } u.$$

The Jacobi elliptic functions may be explicitly defined by first defining $\text{sn } u \equiv$

$\mathrm{sn}(x, k)$ such that

$$\mathrm{sn}(x, k) = Rev\left(\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right).$$

Jacobi's work on elliptic functions led to his well known publication titled *Fundamenta nova theoriae functionum ellipticarum* (1829) in which he presented the four theta functions [55]. The most remarkable achievement of Jacobi was establishing the relation between the elliptic theta functions that are closely related to the elliptic functions [45]. The Jacobi theta functions are the elliptic analogs of the exponential functions which can be expressed in terms of the Jacobi elliptic functions.

Following on from the work of Jacobi on ellliptic functions, the mathematician Karl Weierstrass $(1815 - 1897)$ also made significant contributions to the theory of elliptic functions. His main contributions was introducing new functions which behave in a simpler way. In particular, he replaced the 3 basic types of Jacobi elliptic functions $\mathrm{sn}\ u, \mathrm{cn}\ u$ and $\mathrm{dn}\ u$ by the single function known as the Weierstrass $\wp$ which is also referred to as the (Weierstrass $'P'$) function. Similarly, he replaced the Jacobi theta functions by the sigma and the zeta functions. The Weierstrass elliptic functions are known to parametrize cubics given in the Weierstrass form by

$$(y')^2 = 4x^3 - g_2x - g_3.$$

A third type of elliptic function known as the **Dixon's elliptic functions** was introduced by the British mathematician Alfred Cardew Dixon $(1865 - 1936)$. His main work on elliptic functions is collected in his publication *The elementary properties of the elliptic functions, with examples* (1894) [33]. There are few available research publications on Dixon elliptic functions compared to the Jacobi and Weierstrass elliptic functions that have received considerable research focus to date. The most recent publication on Dixon elliptic functions is the work by Langer & Singer (2013)[65] in which the arc length of a sextic curve referred to as the *trefoil* is expressed in terms of Dixon elliptic functions. A prior publication by Conrad and Flajolet(2006) [26] extensively covers the combinatorial aspects and certain pattern permutations arising from a study of these functions.

The first six terms of the coefficients of the Taylor series expansion of

$$\operatorname{sn}(x,k) = Rev\left(\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right)$$

are

$$\left\{0,1,0,-k^2-1,0,k^4+14k^2+1,0\right\}$$

and for

$$\operatorname{arcsn}(x,k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

are

$$\left\{0,1,0,k^2+1,0,3\left(3k^4+2k^2+3\right),0\right\}.$$

We note that we can find the power series coefficients of $\operatorname{sn}(x,k)$ in the first column of the inverse Riordan array

$$\left[\frac{\operatorname{arcsn}(x,k)}{x}, \operatorname{arcsn}(x,k)\right]^{-1}.$$

More recent work over the last century has focused on aspects of the Taylor series coefficients of elliptic functions such as getting recurrence formulas and combinatorial interpretations with enumerative properties [41, 94, 95, 26, 62]. Elliptic functions have also been studied in relation to their connection to orthogonal polynomials [53, 54]. Using the aid of concrete examples we shall relate elliptic functions to exponential Riordan arrays in the next three chapters. More details pertaining to each of these functions will be elaborated in the subsequent chapters. For the purpose of this thesis the Riordan arrays generated from the Jacobi elliptic functions, Dixonian elliptic functions and Weierstrass elliptic functions respectively will simply be referred to as **Jacobi Riordan arrays, Dixonian Riordan arrays and Weierstrass Riordan arrays** respectively.

1.8.1 Ellipse arc length:

We begin with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Thus

$$\frac{x^2}{a^2} = 1 - \frac{x^2}{y^2} \implies y^2 = b^2\left(1 - \frac{x^2}{a^2}\right).$$

We also get

$$\frac{2xdx}{a^2} + \frac{2ydy}{b^2} = 0$$

Thus,

$$\begin{aligned}
dx^2 + dy^2 &= dx^2 + x^2\frac{dx^2}{y^2}\frac{b^4}{a^4} \\
&= dx^2 + x^2dx^2\frac{b^4}{a^4}\frac{1}{b^2\left(1 - \frac{x^2}{a^2}\right)} \\
&= dx^2 + x^2dx^2\frac{b^2}{a^2}\frac{1}{a^2 - x^2} \\
&= dx^2\left(1 + x^2\frac{b^2}{a^2}\frac{1}{a^2 - x^2}\right) \\
&= dx^2\left(\frac{(a^2 - x^2) + x^2\frac{b^2}{a^2}}{a^2 - x^2}\right) \\
&= dx^2\left(\frac{(1 - \frac{x^2}{a^2}) + x^2\frac{b^2}{a^4}}{1 - \frac{x^2}{a^2}}\right) \\
&= dx^2\frac{1 - \frac{x^2}{a^2}\left(1 - \frac{b^2}{a^2}\right)}{1 - \frac{x^2}{a^2}}
\end{aligned}$$

Thus,

$$\begin{aligned}
s &= \int_0^x \sqrt{dx^2 + dy^2} \\
&= \int_0^x \sqrt{\frac{1 - \frac{x^2}{a^2}\left(1 - \frac{b^2}{a^2}\right)}{1 - \frac{x^2}{a^2}}}dx \\
&= \int_0^x \sqrt{\frac{1 - \frac{x^2}{a^2}\left(\frac{a^2 - b^2}{a^2}\right)}{1 - \frac{x^2}{a^2}}}dx \\
&= a\int_0^{\frac{x}{a}} \sqrt{\frac{1 - k^2t^2}{1 - t^2}}dt
\end{aligned}$$

where $k^2 = \frac{a^2 - b^2}{a^2}$ and we have set $t = \frac{x}{a}$. Note that $0 < k^2 < 1$.

1.9 Elliptic functions derived from the A and Z generating functions of Riordan arrays

In this section the relationship betweeen elliptic functions and the A and Z generating functions of Riordan arrays is established. This relationship serves as the key motivation for further research on elliptic functions and Riordan arrays that has previously not been investigated. We know that a Riordan array can be defined in terms of its generating functions. But an alternative definition of Riordan arrays can be described in terms of its A and Z sequences and their corresponding generating functions [13]. That is for a given pair of generating functions $g(x) \in \mathcal{F}_0$ and $f(x) \in \mathcal{F}_1$ where x is an indeterminate, we can define the Riordan array symbolically as $[g(x), f(x)]$. In terms of the A and Z generating functions associated to the Riordan matrix corresponding to $[g(x), f(x)]$, we can symbolically define the Riordan matrix as $[g_0, \mathrm{A}; \mathrm{Z}]$.

We let $M = [g, f]$ denote an exponential Riordan array. The production matrix of M is the matrix

$$M^{-1}\overline{M},$$

where $\overline{M}$ is the matrix M with the top row removed. The array M determines and is determined by its production matrix. The production matrix has bivariate generating function

$$e^{xy}(\mathrm{Z}_M(x) + y\mathrm{A}_M(x)),$$

where we have

$$\mathrm{A}_M(x) = f'(\bar{f}(x))$$

and

$$\mathrm{Z}_M(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))}.$$

For the inverse exponential Riordan array $M^{-1} = [g, f]^{-1}$, we have

$$\mathrm{A}_{M^{-1}} = \frac{1}{f'(x)},$$

and

$$\mathrm{Z}_{M^{-1}} = -\frac{1}{f'(x)}\frac{g'(x)}{g(x)}.$$

We can express the array $[g, f]$ in terms of $\mathrm{A} = \mathrm{A}_M$ and $\mathrm{Z} = \mathrm{Z}_M$ as follows.

$$[g(x), f(x)] = \left[e^{\int_0^x \mathrm{Z}\left(\mathrm{Rev}\left(\int_0^t \frac{dt}{\mathrm{A}(t)}\right)\right)dt}, \mathrm{Rev}\left(\int_0^x \frac{dt}{\mathrm{A}(t)}\right)\right].$$

Alternatively, we can write

$$[g(x), f(x)] = \left[e^{\int_0^{\mathrm{Rev}\left(\int_0^x \frac{dt}{\mathrm{A}(t)}\right)} \frac{Z(t)}{\mathrm{A}(t)}\,dt}, \mathrm{Rev}\left(\int_0^x \frac{dt}{\mathrm{A}(t)}\right)\right].$$

Now we recall that an integral is called an elliptic integral if it is of the form

$$\int R(x, \sqrt{P(x)})\,dx,$$

where $P(x)$ is a polynomial in x of degree three or four and R is a rational function of its arguments.

Thus if

$$\frac{1}{\mathrm{A}(t)} = R(t, \sqrt{P(t)}),$$

then the above exponential Riordan array can be said to be defined by an elliptic integral.

Elliptic functions are defined as the inverses of elliptic integrals. Thus the expression

$$\mathrm{Rev}\left(\int_0^x \frac{dt}{\mathrm{A}(t)}\right)$$

in the defining relation for $[g, f]$ is an elliptic function when $\frac{1}{\mathrm{A}(t)} = R(t, \sqrt{P(t)})$.

In this case we will have

$$\mathrm{A}(t) = \frac{1}{R(t, \sqrt{P(t)})} = \tilde{R}(t, \sqrt{P(t)}),$$

where

$$\tilde{R} = \frac{1}{R}$$

will also be a rational function.

It is therefore natural to call an exponential Riordan array $M = [g, f]$ an *elliptic Riordan array* if

$$\mathrm{A}_M(t) = R(t, \sqrt{P(t)}),$$

where R is a rational function and $P(t)$ is a polynomial of degree three or four.

Lagrange showed that any elliptic integral can be written in terms of the following three fundamental or normal elliptic integrals.

$$F(x,k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

$$E(x,k) = \int_0^x \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt,$$

$$\Pi(x,\alpha^2,k) = \int_0^x \frac{dt}{(1-\alpha^2 t)\sqrt{(1-t^2)(1-k^2t^2)}}.$$

These integrals are called elliptic integrals of the first, second and third kind, respectively.

As an example of elliptic functions, the Jacobi elliptic functions may be defined by first defining $\mathrm{sn}(x,k)$ as follows,

$$\mathrm{sn}(x,k) = \mathrm{Rev}\left(\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right),$$

involving the elliptic integral of the first kind, and then we define

$$\mathrm{cn}(x,k) = \sqrt{1-\mathrm{sn}(x,k)^2} \quad \text{and} \quad \mathrm{dn}(x,k) = \sqrt{1-k^2\mathrm{sn}(x,k)^2}.$$

Thus if

$$\frac{1}{\mathrm{A}(t)} = \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

i.e.

$$\mathrm{A}(t) = \sqrt{(1-t^2)(1-k^2t^2)},$$

then we obtain an exponential Riordan array with

$$f(x) = \mathrm{sn}(x,k).$$

Chapter 2

Jacobi Elliptic Functions and Riordan Arrays

In this chapter we shall begin by giving an introduction of the Jacobi elliptic functions. This will be followed by various examples of Riordan arrays generated from these functions.

2.1 Types of Jacobi Elliptic Functions and their properties

The initial point to understanding elliptic functions is from the perspective of the two basic trigonometric functions of the sine and cosine. The elliptic function is considered a generalization of these functions since its most basic form reduces to a trigonometric function. The basic properties of the various types of Jacobi elliptic functions can be traced back to the trigonometric sine and cosine functions. The parity property of these functions will later become useful in the construction of a new class of Riordan arrays which is referred here as Jacobi Riordan arrays. The Riordan arrays generated from these functions can be viewed to fall under the Double Riordan group [29] and in particular the checkerboard subgroup of Riordan arrays. The Jacobi elliptic functions were originally derived from the elliptic integral of the first kind [58]. We can define the Jacobi elliptic function sn using a reversion technique, beginning with what will be its reversion such that

$$sn(z,k) = Rev\left(\int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right),$$

which can be rewitten as

$$arcsn(z,k) = \int_0^z \frac{1}{\sqrt{(1-k^2t^2)(1-t^2)}}\,dt.$$

An alternative method for defining $\mathrm{sn}(z,k)$ is to start with the function

$$F(\phi,k) = \int_0^\phi \frac{1}{\sqrt{1-k^2\sin(\theta)}}d\theta.$$

$$\mathrm{arcsn}(z,m) \equiv u(\varphi,m) = \int_0^{\sin\varphi} \frac{dz}{\sqrt{(1-z^2)(1-mz^2)}} \tag{2.1}$$

where $m = k^2$. The parameter k where $0 \le k < 1$ is called the *modulus* of the elliptic integral and φ is called the *amplitude*. The *complementary modulus* is $k' = \sqrt{1-k^2}$. The Legendre form of the elliptic integral which is an alternative definition of elliptic $\mathrm{sn}(z,m)$ is given by

$$\mathrm{arcsn}(z,m) \equiv u(\varphi,m) = \int_0^\varphi \frac{d\theta}{\sqrt{1-m\sin^2\theta}}. \tag{2.2}$$

Equation (2.2) follows from a change of variables which can be determined from (2.1) using the substitution $z = \sin\theta$ such that $dz = \cos z d\theta = \sqrt{1-z^2}d\theta$. We then revert the function F to get the amplitude function

$$\mathrm{am}(u,k) = F^{-1}(u,k) = \phi.$$

Finally we define

$$\mathrm{sn}(u,k) = \sin(\mathrm{am}(u,k)) = \sin(\phi).$$

The two forms of the elliptic integral u in (2.1) & (2.2) are also known as the incomplete elliptic integrals. On the other hand the complete elliptic integrals are given by

$$K(m) := u\left(\frac{\pi}{2},m\right) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-mz^2)}}$$

$$u\left(\frac{\pi}{2}, m\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}$$

with its complete complementary elliptic integral defined as

$$K'(m) := u\left(\frac{\pi}{2}, m'\right) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-(1-m)z^2)}}$$

$$u\left(\frac{\pi}{2}, m'\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-(1-m)\sin^2\theta}}$$

where

$$m + m' = 1 \;\&\; m = (k')^2 \;\&\; \mid m \mid \leq 1 \;\&\; -K < z < K.$$

The hypergeometric interpretation of the complete elliptic integral is defined by

$$K(m) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; m\right) \qquad K'(m) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-m\right).$$

The Jacobi elliptic functions are derived by inverting the elliptic integral (2.2) . The three basic Jacobi elliptic functions from which other forms are obtained are: the elliptic sine sn(u;m)= $\sin\varphi$, the elliptic cosine cn(u;m)= $\cos\varphi$ and the difference function dn(u;k)= $\frac{d\varphi}{du}$. The basic properties of the elliptic Jacobi functions are analogous to those of the well known trigonometric functions. The basic ellliptic functions satisfy the equation

$$\mathrm{sn}^2(u;m) + \mathrm{cn}^2(u;m) = 1$$

from the trigonometric relation $\sin^2\varphi + \cos^2\varphi = 1$. Thus,

$$\mathrm{sn}(u;m) = \sqrt{1-\mathrm{cn}^2(u;m)} \;\&\; \text{dn(u;m)} = \frac{d\varphi}{du} = \sqrt{1-m\mathrm{sn}^2(u;m)}.$$

All three functions $\mathrm{sn}(u,k), \mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$ are doubly periodic. The following holds true for Jacobi elliptic functions:

$$\begin{aligned}
\mathrm{sn}(u+4K,k) &= \mathrm{sn}(u,k),\ \text{where}\ \mathrm{sn}(K,k)=1.\\
\mathrm{sn}(u+2L,k) &= \mathrm{sn}(u,k),\ \text{where}\ \mathrm{cs}(L,k)=i\\
\mathrm{cn}(u+4K,k) &= \mathrm{cn}(u,k).\\
\mathrm{cn}(u+4L,k) &= \mathrm{cn}(u,k).\\
\mathrm{dn}(u+4K,k) &= \mathrm{dn}(u,k).\\
\mathrm{dn}(u+4L,k) &= \mathrm{dn}(u,k).\\
\mathrm{dn}^2(u,k)+k^2\mathrm{sn}^2(u,k) &= 1.
\end{aligned}$$

In addition, the three basic forms of the Jacobi elliptic function determine the other 9 forms of Jacobi elliptic functions such that the definitions are quotients of any of these three. These twelve forms of Jacobi elliptic functions are:

	s	c	d	n
s		sc	sd	sn
c	cs		cd	cn
d	ds	dc		dn
n	ns	nc	nd	

For example $\mathrm{sd}(u;m) = \frac{\mathrm{sn}(u;m)}{\mathrm{dn}(u;m)}$ and similar results apply to each of the other forms shown on the table. The basic properties of the Jacobi elliptic functions are:

- $\mathrm{sn}(0;k) = 0 \qquad \mathrm{sn}(K;k) = 1.$

- $\mathrm{cn}(0;k) = 1 \qquad \mathrm{cn}(K;k) = 0.$

- $\mathrm{dn}(0;k) = 1 \qquad \mathrm{dn}(K;k) = k'.$

In the limit we have,

$$
\begin{aligned}
\lim_{m\to 0} \mathrm{sn}(u,m) &= \sin(u) \\
\lim_{m\to 0} \mathrm{cn}(u,m) &= \cos(u) \\
\lim_{m\to 0} \mathrm{dn}(u,m) &= 1 \\
\lim_{m\to 1} \mathrm{sn}(u,m) &= \tanh(u) \\
\lim_{m\to 1} \mathrm{cn}(u,m) &= \mathrm{sech}(u) \\
\lim_{m\to 1} \mathrm{dn}(u,m) &= \mathrm{sech}(u).
\end{aligned}
$$

The Jacobi elliptic functions satisfy the differential system:

$$
\begin{aligned}
\frac{d}{du}\mathrm{sn}(u,m) &= \mathrm{cn}(u,m)\mathrm{dn}(u,m) \\
\frac{d}{du}\mathrm{cn}(u,m) &= -\mathrm{dn}(u,m)\mathrm{sn}(u,m) \\
\frac{d}{du}\mathrm{dn}(u,m) &= -m\mathrm{sn}(u,m)\mathrm{cn}(u,m)
\end{aligned}
$$

for the initial conditions

$$
\mathrm{sn}(0,m) = 0, \quad \mathrm{cn}(0,m) = 1 \quad \mathrm{dn}(0,m) = 1.
$$

The systems of differential equations satisfied by the Jacobi elliptic functions forms the basis of their applications to some dynamical systems [78].

The plots corresponding to Jacobi sn, Jacobi cn, Inverse Jacobi sn, Inverse Jacobi cn respectively for $m = 1/3$:

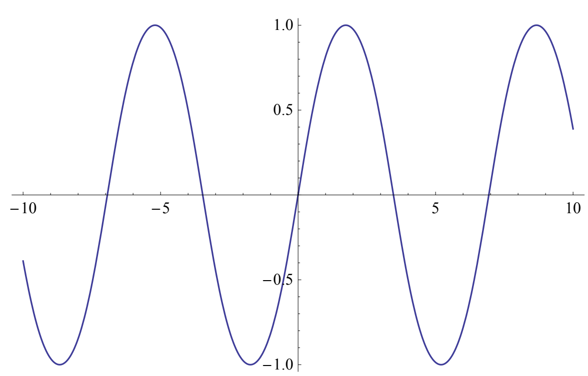


Figure 2.1: A plot of Jacobi sn for $m = 1/3$

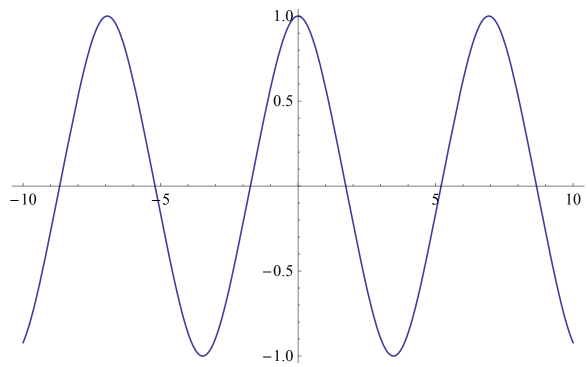


Figure 2.2: A plot of Jacobi cn for $m = 1/3$

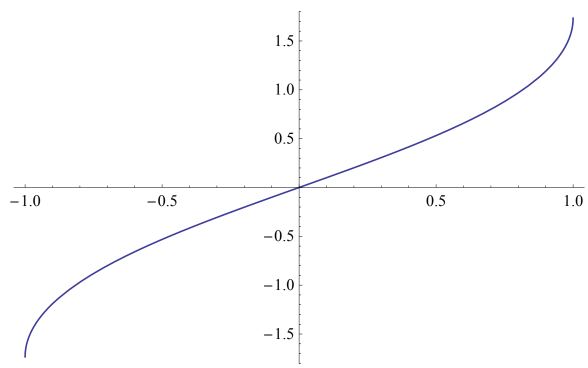


Figure 2.3: A plot of Inverse Jacobi sn for $m = 1/3$

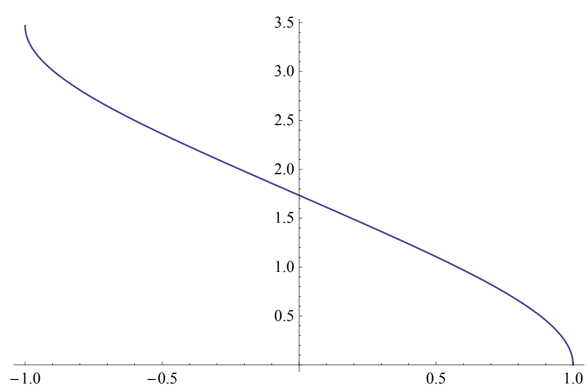


Figure 2.4: A plot of Inverse Jacobi cn for $m = 1/3$

2.2 The Coefficient Arrays of the Jacobi Elliptic Functions

By reviewing the Jacobi elliptic function $arc\mathrm{sn}(z,k)$ in section (2.1), we find that [20, 15]

$$\begin{aligned}
\mathrm{arcsn}(z,k) &= \int_0^z \frac{1}{\sqrt{(1-k^2t^2)(1-t^2)}}\,dt \\
&= \int_0^z \frac{1}{\sqrt{1-(k^2+1)t^2+k^2t^4}}\,dt \\
&= \int_0^z \frac{1}{\sqrt{1-2\frac{k^2+1}{2}\frac{u}{k}+u^2}}\,dt \quad u=kt^2 \\
&= \int_0^z \sum_{n=0}^{\infty} P_n\left(\frac{k^2+1}{2k}\right)u^n\,dt \\
&= \int_0^z \sum_{n=0}^{\infty} P_n\left(\frac{k^2+1}{2k}\right)k^n t^{2n}\,dt \\
&= \sum_{n=0}^{\infty} k^n P_n\left(\frac{1}{2}\left(k+\frac{1}{k}\right)\right)\frac{z^{2n+1}}{2n+1}.
\end{aligned}$$

The first of the power series coefficients for $arc\mathrm{sn}(x,k)$ are therefore given by

$$\left\{0,1,0,\frac{m+1}{6},0,\frac{1}{40}\left(3m^2+2m+3\right),0,\frac{1}{112}\left(5m^3+3m^2+3m+5\right),0,\frac{35m^4+20m^3+18m^2+20m+35}{1152},0,\ldots\right\}$$

where we write $m=k^2$. It is clear from the power series expansion above that $\mathrm{arcsn}(0,k)=0$ for all k, and hence we can revert this power series to obtain the reversion of $\mathrm{arcsn}(x,k)$, which is the Jacobi elliptic function $\mathrm{sn}(x,k)$. The first power series coefficients for $\mathrm{sn}(x,k)$, where we use $m=k^2$, are as follows

$$\left\{0,1,0,\frac{1}{6}(-m-1),0,\frac{1}{120}\left(m^2+14m+1\right),0,\frac{-m^3-135m^2-135m-1}{5040},0,\frac{m^4+1228m^3+5478m^2+1228m+1}{362880},0\right\}.$$

Thus we have

$$\mathrm{sn}(x,k)=x-\frac{1}{3!}(1+k^2)x^3+\frac{1}{5!}(1+14k^2+k^4)x^5-\ldots.$$

We see that $\mathrm{sn}(x,k)$ is an odd function.

We shall spend some time studying these coefficients. Ignoring the zero entries for the moment, we note that the denominator sequence begins

$$1, 6, 120, 5040, 362880, 39916800, 6227020800, 1307674368000, 355687428096000, \ldots.$$

This is the sequence $(2n+1)!$.

We now look at the polynomial sequence in m given by the numerators of the power series coefficients of $\mathrm{sn}(x,k)$. We shall multiply this sequence by $(-1)^n$ to simplify it. We obtain the sequence in m that begins

$$\left\{1, m+1, m^2+14m+1, m^3+135m^2+135m+1, m^4+1228m^3+5478m^2+1228m+1, \ldots\right\}$$

The coefficient array for this sequence of polynomials begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 14 & 1 & 0 & 0 & 0 & 0 \\ 1 & 135 & 135 & 1 & 0 & 0 & 0 \\ 1 & 1228 & 5478 & 1228 & 1 & 0 & 0 \\ 1 & 11069 & 165826 & 165826 & 11069 & 1 & 0 \end{array}\right)$$

We note immediately that this is a symmetric triangle, reminiscent of Pascal's triangle and of the Narayana triangle. Its resemblance in form to the Narayana triangle is not an accident.

We recall that the Narayana triangle has a continued fraction bivariate generating function given by

$$\cfrac{1}{1-x-xy-\cfrac{x^2y}{1-x-xy-\cfrac{x^2y}{1-x-xy-\cfrac{x^2y}{1-\cdots}}}}.$$

It turns out that the above $\mathrm{sn}(x,k)$ inspired triangle has a similar type bivariate generating function

$$\cfrac{1}{1-x-xy-\cfrac{12x^2y}{1-9x-9xy-\cfrac{240x^2y}{1-25x-25xy-\cfrac{1260x^2y}{1-49x-49xy-\cfrac{4032x^2y}{1-\cdots}}}}}.$$

The coefficients in this fraction are based on the sequence

$$1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, \ldots$$

where the numbers are taken in groups of 4 ($1 \cdot 2 \cdot 2 \cdot 3 = 12$ etc) for the "β" coefficients, and two by two for the "α" coefficients ($3 \cdot 3 = 9$ etc).

This J-fraction [25] may also be written equivalently as follows:

$$\cfrac{1}{1-x-\cfrac{xy}{1-\cfrac{12x}{1+\sqrt{\frac{9}{1}}x-\cfrac{9xy}{1-\cfrac{\frac{240}{9}x}{1+\sqrt{\frac{25}{9}}x-\cfrac{25xy}{1-\cfrac{\frac{1260}{25}x}{1+\sqrt{\frac{49}{25}}x-\cfrac{49xy}{1-\cdots}}}}}}}},$$

or

$$\cfrac{1}{1-x-\cfrac{xy}{1-\cfrac{12x}{1+\frac{3}{1}x-\cfrac{9xy}{1-\cfrac{\frac{240}{9}x}{1+\frac{5}{3}x-\cfrac{25xy}{1-\cfrac{\frac{1260}{25}x}{1+\frac{7}{5}x-\cfrac{49xy}{1-\cdots}}}}}}}}.$$

The Hankel transform of the sequence

$$1, m+1, m^2+14m+1, m^3+135m^2+135m+1, \ldots$$

can be calculated to be [25]

$$h_n = k^{n(n+1)} \prod_{i=0}^{2n+1} i!.$$

We note that there is a recurrence for the coefficients $b(n,k) = n!a(n,k)$, where $a(n,k) = [x^n]\mathrm{sn}(x,k)$ which can be described as follows [119].

$$b(n,k) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n = 1, \\ -(k^2+1) & \text{if } n = 3, \\ k^4+2k^2+1 & \text{if } n = 5, \\ f(n) & \text{if } n > 5,\ n \text{ odd} \end{cases}$$

where

$$\begin{aligned}(n-5)f(n) &= (n-1)(1+k^2)b(n-2,k) \\ &+ \sum_{i=2}^{\frac{n-1}{2}} b(2i-1,k)b(n+2-2i,k)\left(\binom{n+1}{2i-1} - 3\binom{n-1}{2i-1}\right) \\ &+ \sum_{i=2}^{\frac{n-1}{2}} (1+k^2)\binom{n-1}{2i-1} b(2i-1,k)b(n-2i,k).\end{aligned}$$

The power series coefficients for $\text{cn}(x,k)$ begin

$$\left\{1, 0, -\frac{1}{2}, 0, \frac{m}{6} + \frac{1}{24}, 0, \frac{1}{720}\left(-16m^2 - 44m - 1\right), 0, \frac{64m^3 + 912m^2 + 408m + 1}{40320}\right\}$$

and so we have

$$\text{cn}(x,k) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}(1+4k^2)x^4 - \frac{1}{6!}(1+44k^2+16k^4)x^6 + \ldots.$$

We see that $\text{cn}(x,k)$ is an even function.

The adjusted sequence of coefficients (multiplying by the denominators multiplying by $(-1)^n$ as appropriate) gives us the polynomial sequence

$$1, -1, 1+4m, -1-44m-16m^2, 1+408m+912m^2+64m^3, -1-3688m-30768m^2-15808m^3-256m^4, \ldots$$

which has the coefficient array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 44 & 16 & 0 & 0 & 0 & 0 \\ 1 & 408 & 912 & 64 & 0 & 0 & 0 \\ 1 & 3688 & 30768 & 15808 & 256 & 0 & 0 \\ 1 & 33212 & 870640 & 1538560 & 259328 & 1024 & 0 \end{pmatrix}$$

This array has the following continued fraction bivariate generating function.

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{4xy}{1-\cfrac{9x}{1-\cfrac{16xy}{1-\cfrac{25x}{1-\cdots}}}}}}.$$

The Hankel transform of the sequence

$$1, -1, 1+4m, -1-44m-16m^2, 1+408m+912m^2+64m^3, -1-3688m-30768m^2-15808m^3-256m^4, \ldots$$

can be calculated to be [25]

$$h_n(m) = m^{\binom{n}{2}} \left(\prod_{k=1}^{n} (2k-2)! \right)^2.$$

The coefficients for the power series of $\mathrm{dn}(u, k)$ begin

$$1, 0, -\frac{m}{2}, 0, \frac{1}{24}\left(m^2+4m\right), 0, \frac{1}{720}\left(-m^3-44m^2-16m\right), 0, \frac{m^4+408m^3+912m^2+64m}{40320}, \ldots$$

and thus we have

$$\mathrm{dn}(x, k) = 1 - \frac{1}{2!}k^2x^2 + \frac{1}{4!}(4k^2+k^4)x^4 - \ldots.$$

The function $\mathrm{dn}(x, k)$ is an even function.

From it we obtain the polynomial sequence

$$1, -m, m^2+4m, -m^3-44m^2-16m, m^4+408m^3+912m^2+64m, \ldots$$

from which (dropping signs) we derive the coefficient array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 16 & 44 & 1 & 0 & 0 & 0 \\ 0 & 64 & 912 & 408 & 1 & 0 & 0 \\ 0 & 256 & 15808 & 30768 & 3688 & 1 & 0 \\ 0 & 1024 & 259328 & 1538560 & 870640 & 33212 & 1 \end{pmatrix}$$

This array has the continued fraction bivariate generating function given by

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{4x}{1-\cfrac{9xy}{1-\cfrac{16x}{1-\cfrac{25xy}{1-\cdots}}}}}},$$

or equivalently,

$$\cfrac{1}{1-xy-\cfrac{4x^2y}{1-4x-9xy-\cfrac{144x^2y}{1-16x-25xy-\cfrac{400x^2y}{1-\ldots}}}}.$$

2.2.1 The Binomial transform in m

The matrix [15]

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 16 & 44 & 1 & 0 & 0 & 0 \\ 0 & 64 & 912 & 408 & 1 & 0 & 0 \\ 0 & 256 & 15808 & 30768 & 3688 & 1 & 0 \\ 0 & 1024 & 259328 & 1538560 & 870640 & 33212 & 1 \end{pmatrix}$$

is more correctly associated to the Jacobi elliptic function dc as it is the coefficient array of

$$(2n)![x^{2n}]\text{dc}(x, -m+1).$$

Multiplying this matrix on the right by the binomial matrix B is the same as letting $m \to m+1$. We get

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 16 & 44 & 1 & 0 & 0 & 0 \\ 0 & 64 & 912 & 408 & 1 & 0 & 0 \\ 0 & 256 & 15808 & 30768 & 3688 & 1 & 0 \\ 0 & 1024 & 259328 & 1538560 & 870640 & 33212 & 1 \end{array}\right) \cdot B =$$

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 6 & 1 & 0 & 0 & 0 & 0 \\ 61 & 107 & 47 & 1 & 0 & 0 & 0 \\ 1385 & 3116 & 2142 & 412 & 1 & 0 & 0 \\ 50521 & 138933 & 130250 & 45530 & 3693 & 1 & 0 \\ 2702765 & 8783986 & 10430983 & 5353260 & 1036715 & 33218 & 1 \end{array}\right).$$

This is the coefficient array associated to the Jacobi elliptic function $\text{dc}(x, -m)$. More precisely, it is the coefficient array for

$$(2n)![x^{2n}]\text{dc}(x, -m).$$

This array has bi-variate generating function given by

$$\cfrac{1}{1-\cfrac{x(y+1)}{1-\cfrac{4x}{1-\cfrac{9x(y+1)}{1-\cfrac{16x}{1-\cfrac{25x(y+1)}{1-\cdots}}}}}}.$$

In like manner, the array

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 6 & 1 & 0 & 0 & 0 & 0 \\ 61 & 107 & 47 & 1 & 0 & 0 & 0 \\ 1385 & 3116 & 2142 & 412 & 1 & 0 & 0 \\ 50521 & 138933 & 130250 & 45530 & 3693 & 1 & 0 \\ 2702765 & 8783986 & 10430983 & 5353260 & 1036715 & 33218 & 1 \end{array}\right) \cdot B =$$

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 12 & 8 & 1 & 0 & 0 & 0 & 0 \\ 216 & 204 & 50 & 1 & 0 & 0 & 0 \\ 7056 & 8640 & 3384 & 416 & 1 & 0 & 0 \\ 368928 & 550800 & 289008 & 60312 & 3698 & 1 & 0 \\ 28340928 & 50018688 & 33043248 & 9832320 & 1202820 & 33224 & 1 \end{array}\right),$$

is the array associated to

$$(2n)![x^{2n}]\mathrm{dc}(x, -m-1),$$

with bi-variate generating function

$$\cfrac{1}{1-\cfrac{x(y+2)}{1-\cfrac{4x}{1-\cfrac{9x(y+2)}{1-\cfrac{16x}{1-\cfrac{25x(y+2)}{1-\cdots}}}}}}.$$

In a similar manner, the array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 14 & 1 & 0 & 0 & 0 & 0 \\ 1 & 135 & 135 & 1 & 0 & 0 & 0 \\ 1 & 1228 & 5478 & 1228 & 1 & 0 & 0 \\ 1 & 11069 & 165826 & 165826 & 11069 & 1 & 0 \\ 1 & 99642 & 4494351 & 13180268 & 4494351 & 99642 & 1 \end{pmatrix}$$

is given by

$$n!(-1)^n[x^{2n+1}]\text{sn}(x,m),$$

with bivariate generating function

$$\cfrac{1}{1-x-\cfrac{xy}{1-\cfrac{12x}{1+\sqrt{\frac{9}{1}}x-\cfrac{9xy}{1-\cfrac{\frac{240}{9}x}{1+\sqrt{\frac{25}{9}}x-\cfrac{25xy}{1-\cfrac{\frac{1260}{25}x}{1+\sqrt{\frac{49}{25}}x-\cfrac{49xy}{1+\sqrt{\frac{81}{49}}x-\cdots}}}}}}}.$$

Then the array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 14 & 1 & 0 & 0 & 0 & 0 \\ 1 & 135 & 135 & 1 & 0 & 0 & 0 \\ 1 & 1228 & 5478 & 1228 & 1 & 0 & 0 \\ 1 & 11069 & 165826 & 165826 & 11069 & 1 & 0 \\ 1 & 99642 & 4494351 & 13180268 & 4494351 & 99642 & 1 \end{pmatrix} \cdot B =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 16 & 16 & 1 & 0 & 0 & 0 & 0 \\ 272 & 408 & 138 & 1 & 0 & 0 & 0 \\ 7936 & 15872 & 9168 & 1232 & 1 & 0 & 0 \\ 353792 & 884480 & 729728 & 210112 & 11074 & 1 & 0 \\ 22368256 & 67104768 & 71997696 & 32154112 & 4992576 & 99648 & 1 \end{pmatrix}$$

has bivariate generating function

$$\cfrac{1}{1-x-\cfrac{x(y+1)}{1-\cfrac{12x}{1+3x-\cfrac{9x(y+1)}{1-\cfrac{\frac{240}{9}x}{1+\frac{5}{3}x-\cfrac{25x(y+1)}{1-\cfrac{\frac{1260}{25}x}{1+\frac{7}{5}x-\cfrac{49x(y+1)}{1-\cdots}}}}}}}}.$$

It is the array for

$$n!(-1)^n[x^{2n+1}]\mathrm{sn}(x,m+1).$$

2.3 Jacobi Riordan arrays

In this section Riordan arrays are constructed based on the Jacobi elliptic functions in terms of the elliptic modulus m under their various subgroups. A combinatorial interpretaion for each of these matrices corresponding to their trigonometric and hyperbolic generating functions will be noted using the **OEIS** online resource. The possible combinatorial interpretations of the Taylor series coefficients of the Jacobi elliptic functions sn and cn have been investigated in [41, 94, 95, 119]. Using the Taylor series coefficients of these functions we shall proceed with the construction of Riordan matrices with their corresponding production matrices to get an alternative view in the study of such functions.

2.3.1 Mixed subgroup of Jacobi Riordan arrays

2.3.1.1 $[\mathbf{cn}(z,m), \mathbf{sn}(z,m)]$

The coefficient matrix of $[\mathrm{cn}(z,m), \mathrm{sn}(z,m)]$ begins

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6\left(-\frac{m}{6}-\frac{2}{3}\right) & 0 & 1 & 0 & 0 & 0 \\ 24\left(\frac{m}{6}+\frac{1}{24}\right) & 0 & 12\left(-\frac{m}{3}-\frac{5}{6}\right) & 0 & 1 & 0 & 0 \\ 0 & m^2+44m+16 & 0 & 20\left(-\frac{m}{2}-1\right) & 0 & 1 & 0 \\ -16m^2-44m-1 & 0 & 16m^2+224m+91 & 0 & 30\left(-\frac{2m}{3}-\frac{7}{6}\right) & 0 & 1 \end{pmatrix}$$

which is equivalent to

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -m-4 & 0 & 1 & 0 & 0 & 0 \\ 4m+1 & 0 & -2(2m+5) & 0 & 1 & 0 & 0 \\ 0 & m^2+44m+16 & 0 & -10(m+2) & 0 & 1 & 0 \\ -16m^2-44m-1 & 0 & 16m^2+224m+91 & 0 & -5(4m+7) & 0 & 1 \end{pmatrix}.$$

Remark:

- The row sums for $m = 0$ form the sequence $(1, 1, 0, -3, -8, -3, 56, ...)$ which has e.g.f $\cos(z)e^{\sin(z)}$.
- The row sums for $m = 1$ form the sequence $(1, 1, 0, -4, -8, 32, 216, ...)$ corresponds to **A009265** with e.g.f $\frac{e^{\tanh(z)}}{\cosh(z)}$.

The production matrix of A in terms of m :

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -m-3 & 0 & 1 & 0 & 0 \\ 3(m-1) & 0 & -3(m+2) & 0 & 1 & 0 \\ 0 & -3\left(m^2-6m+5\right) & 0 & -2(3m+5) & 0 & 1 \\ 15\left(m^2+2m-3\right) & 0 & -15\left(m^2-4m+3\right) & 0 & -5(2m+3) & 0 \end{pmatrix}.$$

If $m = 0, -1, 1$ then $[\mathrm{cn}(z,m), \mathrm{sn}(z,m)]$ produces the Riordan arrays

$$D = \{[\cos(z), \sin(z)], [\mathrm{cn}(z,-1), \mathrm{sn}(z,-1)], [\mathrm{sech}(z), \tanh(z)]\} \text{ respectively.}$$

The production matrices from C for m=-1,0,1 associated to the Riordan arrays in D are as follows:

$$E = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ -6 & 0 & -3 & 0 & 1 \\ 0 & -36 & 0 & -4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ -3 & 0 & -6 & 0 & 1 \\ 0 & -15 & 0 & -10 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -4 & 0 & 1 & 0 \\ 0 & 0 & -9 & 0 & 1 \\ 0 & 0 & 0 & -16 & 0 \end{pmatrix} \right\}.$$

Remark: The production matrix in E at $m = 1$ is tridiagonal which indicates that the inverse of $[\mathrm{cn}(z,1), \mathrm{sn}(z,1)] = [\mathrm{sech}(z), \tanh(z)]^{-1} = \left[\frac{1}{\sqrt{1-z^2}}, \tanh^{-1}(z)\right]$ is the coefficient array of a family of orthogonal polynomials. The three term recurrence relation for the family of orthogonal polynomials is given by

$$P_{n+1}(z) = zP_n(z) + n^2 P_{n-1}(z),\ \forall n \geq 1$$

with $P_0(z) = 1,\ P_1(z) = z$.

$$\text{In particular, let } Q_n(z) = \frac{P_n(iz)}{i^n},\ (i^2 = -1)$$

we get

$$Q_{n+1}(z) = zQ_n(z) - n^2 Q_{n-1}(z),\ \forall n \geq 1.$$

The generating functions of the r and c sequences associated to the production matrices of D and C are listed below:

$$\begin{aligned}
\mathrm{r}(z,m) &= \mathrm{cn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right) \mathrm{dn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right) \\
\mathrm{r}(z,1) &= 1 - z^2 \\
\mathrm{r}(z,0) &= \sqrt{1-z^2} \\
\mathrm{c}(z,m) &= \frac{z\mathrm{cn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right) \mathrm{dn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right)}{z^2 - 1} \\
\mathrm{c}(z,1) &= -z \\
\mathrm{c}(z,0) &= -\frac{z}{\sqrt{1-z^2}}.
\end{aligned}$$

Alternatively, we can derive the r and c generating function of $[cn(z,m), sn(z,m)]$ using the elliptic integrals defined in (2.1) as follows:

$$\mathrm{c}(z) \equiv \mathrm{A}(z) = \mathrm{sn}'(\overline{\mathrm{sn}}(z,m))\ \&\ \mathrm{Z}(z) = \frac{\mathrm{cn}'(\overline{\mathrm{sn}}(z,m))}{\mathrm{cn}(\overline{\mathrm{sn}}(z,m))}$$

For

$$\begin{aligned}
\mathrm{A}(z) &= \mathrm{sn}'(\overline{\mathrm{sn}}(z,m)) \\
&= \mathrm{sn}'\left(\int_0^z \frac{1}{\sqrt{(1-mu^2)(1-u^2)}}du\right) \\
&= \mathrm{cn}(\overline{\mathrm{sn}}(z,m))\mathrm{dn}(\overline{\mathrm{sn}}(z,m)) \\
&= (\sqrt{1-\mathrm{sn}^2(\overline{\mathrm{sn}}(z,m))})(\sqrt{1-m\mathrm{sn}^2(\overline{\mathrm{sn}}(z,m))}) \\
&= (\sqrt{1-z^2})(\sqrt{1-mz^2}).
\end{aligned}$$

For

$$\begin{aligned}
\mathrm{r}(z) \equiv \mathrm{Z}(z) &= \frac{\mathrm{cn}'(\overline{\mathrm{sn}}(z,m))}{\mathrm{cn}(\overline{\mathrm{sn}}(z,m)))} \\
&= \frac{-\mathrm{sn}(\overline{\mathrm{sn}}(z,m))\mathrm{dn}(\overline{\mathrm{sn}}(z,m))}{\mathrm{cn}(\overline{\mathrm{sn}}(z,m))} \\
&= \frac{-z\sqrt{1-mz^2}}{\sqrt{1-z^2}}.
\end{aligned}$$

2.3.1.2 $[\mathbf{cd}(z,m), \mathbf{sn}(z,m)]$

The coefficient matrix of $[\mathrm{cd}(z,m), \mathrm{sn}(z,m)]$ is given by

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
m-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2(m-2) & 0 & 1 & 0 & 0 & 0 \\
5m^2-6m+1 & 0 & 2(m-5) & 0 & 1 & 0 & 0 \\
0 & 16\left(m^2-m+1\right) & 0 & -20 & 0 & 1 & 0 \\
61m^3-107m^2+47m-1 & 0 & 31m^2+14m+91 & 0 & -5(m+7) & 0 & 1
\end{pmatrix}.$$

Remark:

- The row sums of A for $m=0$ form the sequence $(1,1,0,-3,-8,-3,56,...)$ which has the e.g.f $\cos(z)e^{\sin(z)}$.

- The row sums of A for $m=1$ form the sequence $(1,1,1,-1,-7,-3,97,...)$ corresponds to **A003723** with e.g.f $e^{\tanh(z)}$.

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ m-1 & 0 & 1 & 0 & 0 & 0 \\ 0 & m-3 & 0 & 1 & 0 & 0 \\ 3\left(m^2-1\right) & 0 & -6 & 0 & 1 & 0 \\ 0 & 9m^2+6m-15 & 0 & -2(m+5) & 0 & 1 \\ 15\left(3m^3-m^2+m-3\right) & 0 & 15\left(m^2+2m-3\right) & 0 & -5(m+3) & 0 \end{pmatrix}.$$

If $m = -1, 0, 1$ then $[\mathrm{cd}(z,m), sn(z,m)]$ produces the Riordan arrays: $C = \{[-1, \mathrm{sn}(z,-1)], [\cos(z), \sin(z)], [1, \tanh(z)]$ respectively. The production matrices from B in terms of $m = -1, 0, 1$ corresponding to the Riordan arrays in C respectively are as follows:

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & -4 & 0 & 1 & 0 \\ 0 & 0 & -6 & 0 & 1 \\ 0 & -12 & 0 & -8 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ -3 & 0 & -6 & 0 & 1 \\ 0 & -15 & 0 & -10 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & -6 & 0 & 1 \\ 0 & 0 & 0 & -12 & 0 \end{pmatrix} \right\}.$$

Remark: The production matrix in D at $m = 1$ is tridiagonal which indicates that the inverse of $[\mathrm{cn}(z,1), \mathrm{sn}(z,1)] = [1, \tanh(z)]^{-1} = \left[1, \tanh^{-1}(z)\right]$ is the coefficient array of a family of orthogonal polynomials.

The generating functions of the r and c sequences associated to the production matrices of B and D are listed below:

$$\begin{aligned} r(z,m) &= \mathrm{cn}\left(\mathrm{sn}^{-1}(z|m)\middle| m\right) \mathrm{dn}\left(\mathrm{sn}^{-1}(z|m)\middle| m\right) \\ r(z,1) &= 1 - z^2 \\ r(z,0) &= \sqrt{1-z^2} \\ c(z,m) &= \frac{(m-1)\left(mz^2-1\right) \mathrm{cd}\left(\mathrm{sn}^{-1}(z|m)\middle| m\right) \mathrm{nd}\left(\mathrm{sn}^{-1}(z|m)\middle| m\right) \mathrm{sd}\left(\mathrm{sn}^{-1}(z|m)\middle| m\right)}{z^2-1} \\ c(z,1) &= 0 \\ c(z,0) &= -\frac{z}{\sqrt{1-z^2}}. \end{aligned}$$

2.3.1.3 $[\mathbf{cd}(z,m), \mathbf{sc}(z,m)]$

The coefficient matrix of $[\mathrm{cd}(z,m), \mathrm{sc}(z,m)]$ is given by:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ m-1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2m-1 & 0 & 1 & 0 & 0 & 0 \\ 5m^2-6m+1 & 0 & 2(m+1) & 0 & 1 & 0 & 0 \\ 0 & 16m^2-16m+1 & 0 & 10 & 0 & 1 & 0 \\ 61m^3-107m^2+47m-1 & 0 & 31m^2-46m+31 & 0 & -5(m-5) & 0 & 1 \end{pmatrix}.$$

- The row sums of A for $m = 0$ form the sequence $M = (1, 1, 0, 0, 4, 12, 56, ...)$ which has e.g.f $\cos(z)e^{\tan(z)}$. If M is multiplied with -1^n such that $n = 0, 1, 2, 3, ...$ is the index of M then $(-1)^n M$ corresponds to **A009114** with e.g.f $\frac{\cos(z)}{e^{\tan(z)}}$.
- The row sums of A for $m = 1$ form the sequence $(1, 1, 1, 2, 5, 12, 37, ...)$ which corresponds to **A003724** with e.g.f $e^{\sinh(z)}$.

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ m-1 & 0 & 1 & 0 & 0 & 0 \\ 0 & m & 0 & 1 & 0 & 0 \\ 3(m-1)m & 0 & 3 & 0 & 1 & 0 \\ 0 & 3m(3m-4) & 0 & 8-2m & 0 & 1 \\ 15m\left(3m^2-7m+4\right) & 0 & 15(m-2)m & 0 & -5(m-3) & 0 \end{pmatrix}.$$

If $m = -1, 0, 1$ then $[\mathrm{cd}(z, m), \mathrm{sc}(z, m]$ produces the Riordan arrays:

$$C = \{[\mathrm{cd}(z, -1), \mathrm{sc}(z, -1)], [\cos(z), \tan(z)], [1, \sinh(z)]\} \text{ respectively.}$$

The production matrix of C for the case $m = -1, 0, 1$ is given by

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 6 & 0 & 3 & 0 & 1 \\ 0 & 21 & 0 & 10 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 8 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & -3 & 0 & 6 & 0 \end{pmatrix} \right\}.$$

The r and c generating functions of the production matrix of B and D are listed as follows:

$$\begin{aligned} r(z, m) &= \left(z^2+1\right) \mathrm{dn}\left(\mathrm{sc}^{-1}(z|m)\middle| m\right) \\ r(z, 1) &= \sqrt{z^2+1} \\ c(z, m) &= \frac{(m-1)\mathrm{cd}\left(\mathrm{sc}^{-1}(z|m)\middle| m\right) \mathrm{nd}\left(\mathrm{sc}^{-1}(z|m)\middle| m\right) \mathrm{sd}\left(\mathrm{sc}^{-1}(z|m)\middle| m\right)}{1-(m-1)z^2} \\ c(z, 1) &= 0. \end{aligned}$$

2.3.1.4 $[\mathbf{cd}(z,m),\mathbf{sd}(z,m)]$

The coefficient matrix of $[\mathrm{cd}(z,m),\mathrm{sd}(z,m)]$ is given by

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ m-1 & 0 & 1 & 0 & 0 & 0\\ 0 & 6\left(\frac{5m}{6}-\frac{2}{3}\right) & 0 & 1 & 0 & 0\\ 5m^2-6m+1 & 0 & 12\left(\frac{7m}{6}-\frac{5}{6}\right) & 0 & 1 & 0\\ 0 & 61m^2-76m+16 & 0 & 20\left(\frac{3m}{2}-1\right) & 0 & 1 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ m-1 & 0 & 1 & 0 & 0 & 0\\ 0 & 5m-4 & 0 & 1 & 0 & 0\\ 5m^2-6m+1 & 0 & 2(7m-5) & 0 & 1 & 0\\ 0 & 61m^2-76m+16 & 0 & 30m-20 & 0 & 1 \end{pmatrix}.$$

Remark

- The numbers $1,6,12,20,30,...$ positioned along the $(n+2)$ diagonal of the matrix A corresponds to **A180291**.
- The row sums of A for $m=0$ form the sequence $(1,1,0,-3,-8,-3,56,...)$ which has the e.g.f $\cos(z)e^{\sin(z)}$.
- The row sums of A for $m=1$ form the sequence $(1,1,1,2,5,12,37,...)$ corresponds to **A003724** with e.g.f $e^{\sinh(z)}$.

The production matrix of A in terms of m is given by

$$B=\begin{pmatrix} 0 & 1 & 0 & 0\\ m-1 & 0 & 1 & 0\\ 0 & 4m-3 & 0 & 1\\ 3m-3 & 0 & 9m-6 & 0 \end{pmatrix}.$$

If $m=-1,0,1$ then $[\mathrm{cd}(z,m),\mathrm{sd}(z,m)]$ produces the Riordan arrays:

$$C=\{[\mathrm{cd}(z,-1),\mathrm{sd}(z,-1)],[\cos(z),\sin(z)],[1,\sinh(z)]\}\text{ respectively.}$$

The production matrices of B corresponding to the Riordan arrays C if $m=-1,0,1$ respectively are given by

$$D=\left\{\begin{pmatrix} 0 & 1 & 0 & 0\\ -2 & 0 & 1 & 0\\ 0 & -7 & 0 & 1\\ -6 & 0 & -15 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & -3 & 0 & 1\\ -3 & 0 & -6 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 3 & 0 \end{pmatrix}\right\}.$$

The r and c generating functions of the production matrices B and D are listed below.

$$\begin{aligned}
r(z,m) &= \operatorname{cd}\left(\operatorname{sd}^{-1}(z|m)\middle| m\right)\operatorname{nd}\left(\operatorname{sd}^{-1}(z|m)\middle| m\right)\\
r(z,1) &= \sqrt{z^2+1}\\
r(z,0) &= \sqrt{1-z^2}\\
c(z,m) &= \frac{(m-1)z\operatorname{cd}\left(\operatorname{sd}^{-1}(z|m)\middle| m\right)\operatorname{nd}\left(\operatorname{sd}^{-1}(z|m)\middle| m\right)}{(m-1)z^2+1}\\
c(z,1) &= 0\\
c(z,0) &= -\frac{z}{\sqrt{1-z^2}}.
\end{aligned}$$

2.3.1.5 $[\mathbf{dn}(z,m),\mathbf{sn}(z,m)]$

The coefficient matrix of $[\operatorname{dn}(z,m),\operatorname{sn}(z,m)]$ is given by

$$A=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0\\
-m & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 6\left(-\frac{2m}{3}-\frac{1}{6}\right) & 0 & 1 & 0 & 0 & 0\\
m^2+4m & 0 & 12\left(-\frac{5m}{6}-\frac{1}{3}\right) & 0 & 1 & 0 & 0\\
0 & 16m^2+44m+1 & 0 & 20\left(-m-\frac{1}{2}\right) & 0 & 1 & 0\\
-m^3-44m^2-16m & 0 & 91m^2+224m+16 & 0 & 30\left(-\frac{7m}{6}-\frac{2}{3}\right) & 0 & 1
\end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0\\
-m & 0 & 1 & 0 & 0 & 0 & 0\\
0 & -4m-1 & 0 & 1 & 0 & 0 & 0\\
m(m+4) & 0 & -2(5m+2) & 0 & 1 & 0 & 0\\
0 & 16m^2+44m+1 & 0 & -10(2m+1) & 0 & 1 & 0\\
-m\left(m^2+44m+16\right) & 0 & 91m^2+224m+16 & 0 & -5(7m+4) & 0 & 1
\end{pmatrix}.$$

Remark

- The numbers $1,6,12,20,30,\ldots$ positioned along the $n+2,n$ diagonal of the matrix A corresponds to **A180291.**

- The row sums of A for $m=0$ form the sequence $(1,1,1,0,-3,-8,-3,\ldots)$ which correspond to **A002017** with e.g.f $e^{\sin(z)}$.

- The row sums of A for $m=1$ form the sequence $(1,1,0,-4,-8,32,216,\ldots)$ which corresponds to **A009265** with e.g.f $\frac{e^{\tanh(z)}}{\cosh(z)}$.

The production matrix of A in terms of m is given by:

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -m & 0 & 1 & 0 & 0 & 0 \\ 0 & -3m-1 & 0 & 1 & 0 & 0 \\ -3(m-1)m & 0 & -6m-3 & 0 & 1 & 0 \\ 0 & -3\left(5m^2-6m+1\right) & 0 & -2(5m+3) & 0 & 1 \\ 15m\left(-3m^2+2m+1\right) & 0 & -15\left(3m^2-4m+1\right) & 0 & -5(3m+2) & 0 \end{pmatrix}.$$

If $m = -1, 0, 1$ then $[\mathrm{dn}(z,m), \mathrm{sn}(z,m)]$ produces the Riordan arrays

$C = \{[\mathrm{dn}(z,-1), \mathrm{sn}(z,-1)], [1, \sin(z)], [\mathrm{sech}(z), \tanh(z)]\}$ respectively.

The production matrices of B associated to the Riordan arrays in C for $m = -1, 0, 1$ are as follows:

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ -6 & 0 & 3 & 0 & 1 \\ 0 & -36 & 0 & 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 & 1 \\ 0 & -3 & 0 & -6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -4 & 0 & 1 & 0 \\ 0 & 0 & -9 & 0 & 1 \\ 0 & 0 & 0 & -16 & 0 \end{pmatrix} \right\}.$$

Remark: The tri-diagonal matrix for $m = 1$ in D indicates that the inverse of the Riordan array $[\mathrm{sech}(z), \tanh(z)]$ forms the coefficient array of a family of orthogonal polynomials. This is **A060524** the number of degree $n-$ permutations with k odd cycles. $\left[\frac{1}{\sqrt{1-z^2}}, \tanh^{-1}(z)\right]$ is the coefficient array of a family of orthogonal polynomials. The three term recurrence relation for the family of orthogonal polynomials is given by

$$P_{n+1}(z) = zP_n(z) + n^2 P_{n-1}(z),\ \forall n \geq 1$$

with $P_0(z) = 1,\ P_1(z) = z$.

$$\text{In particular, let } Q_n(z) = \frac{P_n(iz)}{i^n},\ (i^2 = -1)$$

we get

$$Q_{n+1}(z) = zQ_n(z) - n^2 Q_{n-1}(z),\ \forall n \geq 1.$$

The generating functions of the r and c sequences associated to the production matrices of B and D are listed below:

$$\begin{aligned} r(z,m) &= \operatorname{cn}\left(\operatorname{sn}^{-1}(z,m)\middle|\, m\right) \operatorname{dn}\left(\operatorname{sn}^{-1}(z|m)\middle|\, m\right) \\ &= (\sqrt{1-z^2})(\sqrt{1-mz^2}) \\ &= \sqrt{(1-z^2)(1-mz^2)}. \\ r(z,1) &= 1-z^2. \\ r(z,0) &= \sqrt{1-z^2}. \\ c(z,m) &= \frac{-m\operatorname{sn}(\bar{\operatorname{sn}}(z,m),m)\operatorname{cn}(\bar{\operatorname{sn}}(z,m),m)}{\operatorname{dn}(\bar{\operatorname{sn}}(z,m),m)} \\ &= \frac{-mz\sqrt{1-z^2}}{\sqrt{1-mz^2}}. \\ c(z,1) &= -z \\ c(z,0) &= 0. \end{aligned}$$

2.3.1.6 $[\mathbf{dc}(z,m), \mathbf{sn}(z,m)]$

The coefficient matrix of $[\operatorname{dc}(z,m), \operatorname{sn}(z,m)]$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1-m & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6\left(\frac{1}{3}-\frac{2m}{3}\right) & 0 & 1 & 0 & 0 & 0 \\ m^2-6m+5 & 0 & 12\left(\frac{1}{6}-\frac{5m}{6}\right) & 0 & 1 & 0 & 0 \\ 0 & 16\left(m^2-m+1\right) & 0 & -20m & 0 & 1 & 0 \\ -m^3+47m^2-107m+61 & 0 & 91m^2+14m+31 & 0 & 30\left(-\frac{7m}{6}-\frac{1}{6}\right) & 0 & 1 \end{pmatrix}$$

which is equivalent to

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1-m & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2-4m & 0 & 1 & 0 & 0 & 0 \\ m^2-6m+5 & 0 & 2-10m & 0 & 1 & 0 & 0 \\ 0 & 16\left(m^2-m+1\right) & 0 & -20m & 0 & 1 & 0 \\ -m^3+47m^2-107m+61 & 0 & 91m^2+14m+31 & 0 & -5(7m+1) & 0 & 1 \end{pmatrix}.$$

Remark

- The numbers $1, 6, 12, 20, 30, ...$ positioned along the $n+2, n$ diagonal of the matrix A correspond to **A180291.**
- The row sums of A for $m = 0$ form the sequence $(1, 1, 2, 3, 8, 17, 88, ...)$ corresponding to **A009207** with e.g.f $\frac{e^{\sin(z)}}{\cos(z)}$.

- The row sums of A for $m = 1$ form the sequence $(1, 1, 1, -1, -7, -3, 97, ...)$ corresponding to **A003723** having e.g.f $e^{\tanh(z)}$.

The production matrix of $[\mathrm{dc}(z, m), \mathrm{sn}(z, m)]$ in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1-m & 0 & 1 & 0 & 0 & 0 \\ 0 & 1-3m & 0 & 1 & 0 & 0 \\ 3-3m^2 & 0 & -6m & 0 & 1 & 0 \\ 0 & -15m^2+6m+9 & 0 & -2(5m+1) & 0 & 1 \\ -15\left(3m^3-m^2+m-3\right) & 0 & -45m^2+30m+15 & 0 & -5(3m+1) & 0 \end{pmatrix}.$$

If $m = -1, 0, 1$ then $[\mathrm{dc}(z, m), \mathrm{sn}(z, m)]$ produces the Riordan arrays

$$C = \{[\mathrm{dc}(z, -1), \mathrm{sn}(z|-1)], [\sec(z), \sin(z)], [1, \tanh(z)], \} \text{ respectively.}$$

The production matrices of B associated to the Riordan arrays C if $m = -1, 0, 1$ are as follows respectively:

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 1 \\ 0 & -12 & 0 & 8 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 0 & 9 & 0 & -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & -6 & 0 & 1 \\ 0 & 0 & 0 & -12 & 0 \end{pmatrix} \right\}.$$

Remark: The tri-diagonal matrix for $m = 1$ in D indicates that the inverse of the Riordan array $[1, \tanh(z)]$ forms the coefficient array of a family of orthogonal polynomial sequence. This is **A111594** the triangle of arctanh numbers. The r and c sequences corresponding of the production matrices of B and D are listed as follows:

$$\begin{aligned} r(z,m) &= \mathrm{cn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right)\mathrm{dn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right) \\ r(z,1) &= 1-z^2 \\ r(z,0) &= \sqrt{1-z^2} \\ c(z,m) &= \frac{(m-1)z\,\mathrm{cn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right)\mathrm{dn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right)}{\left(z^2-1\right)\left(mz^2-1\right)} \\ c(z,1) &= 1 \\ c(z,0) &= -\frac{z}{\sqrt{1-z^2}}. \end{aligned}$$

2.3.1.7 [**cn**(z, m), **am**(z, m)]

The coefficient matrix of $[\mathrm{cn}(z, m), \mathrm{am}(z, m)]$ is given by:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6\left(-\frac{m}{6}-\frac{1}{2}\right) & 0 & 1 & 0 & 0 & 0 \\ 24\left(\frac{m}{6}+\frac{1}{24}\right) & 0 & 12\left(-\frac{m}{3}-\frac{1}{2}\right) & 0 & 1 & 0 & 0 \\ 0 & m^2+34m+5 & 0 & 20\left(-\frac{m}{2}-\frac{1}{2}\right) & 0 & 1 & 0 \\ -16m^2-44m-1 & 0 & 16m^2+144m+15 & 0 & 30\left(-\frac{2m}{3}-\frac{1}{2}\right) & 0 & 1 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -m-3 & 0 & 1 & 0 & 0 & 0 \\ 4m+1 & 0 & -4m-6 & 0 & 1 & 0 & 0 \\ 0 & m^2+34m+5 & 0 & -10(m+1) & 0 & 1 & 0 \\ -16m^2-44m-1 & 0 & 16m^2+144m+15 & 0 & -5(4m+3) & 0 & 1 \end{pmatrix}.$$

Remark

- The numbers $1, 6, 12, 20, 30, ...$ positioned along the $n+2, n$ diagonal of the matrix A correspond to **A180291.**
- The row sums of A for $m = 0$ form the sequence $(1, 1, 0, -2, -4, -4, 0, ...)$ correspond to **A146559** with o.g.f $e^z \cos(z)$.
- The row sums of A for $m = 1$ form the sequence $(1, 1, 0, -3, -4, 21, 80, ...)$ corresponds to **A012123** with e.g.f $e^{(\sin^{-1}(\tanh(z)))} = e^{gdz}$ (**Gudermannian function**).

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -m-2 & 0 & 1 & 0 & 0 \\ 3m-2 & 0 & -3(m+1) & 0 & 1 & 0 \\ 0 & -3m^2+16m-8 & 0 & -6m-4 & 0 & 1 \\ 15m^2-16 & 0 & -5\left(3m^2-10m+4\right) & 0 & -5(2m+1) & 0 \end{pmatrix}.$$

If $m = 0, 1$ then $[\text{cn}(z, m), \text{am}(z, m)]$ produces the Riordan arrays:

$$\{[\cos(z), z], \left[\text{sech}(z), \sin^{-1}(\tanh(z))\right]\}.$$

The matrix $\left[\text{sech}(z), \sin^{-1}(\tanh(z))\right]$ corresponds to **A147308** . The inverse matrix of $\left[\text{sech}(z), \sin^{-1}(\tanh(z))\right]$ is $[\sec(z), \log(\sec(z) + \tan(z))]$ which corresponds to **A147309**. Its row sums are the once shifted Euler up/down numbers

(**A000111**) . The production matrices derived from B if $m = -1, 0, 1$ are respectively as follows:

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -5 & 0 & 0 & 0 & 1 \\ 0 & -27 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ -2 & 0 & -3 & 0 & 1 \\ 0 & -8 & 0 & -4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 1 & 0 & -6 & 0 & 1 \\ 0 & 5 & 0 & -10 & 0 \end{pmatrix} \right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D are listed as follows:

$$\begin{aligned} r(z,m) &= \operatorname{dn}(F(z|m)|m) \\ r(z,1) &= \operatorname{sech}(F(z|1)) \\ c(z,m) &= -\frac{\operatorname{dn}(F(z|m)|m)\operatorname{sn}(F(z|m)|m)}{\operatorname{cn}(F(z|m)|m)} \\ c(z,1) &= -\tanh(F(z|1)) \\ c(z,0) &= -\tan(z). \end{aligned}$$

2.3.2 Appell subgroup of Jacobi Riordan arrays

Recall from (1.4.1) that the elements of the Appell subgroup are of the form $[d(z), z]$.

2.3.2.1 $[\mathbf{cd}(z,m), z]$

The coefficient matrix of $[\operatorname{cd}(z,m), z]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ m-1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3(m-1) & 0 & 1 & 0 & 0 & 0 \\ 5m^2-6m+1 & 0 & 6(m-1) & 0 & 1 & 0 & 0 \\ 0 & 5\left(5m^2-6m+1\right) & 0 & 10(m-1) & 0 & 1 & 0 \\ 61m^3-107m^2+47m-1 & 0 & 15\left(5m^2-6m+1\right) & 0 & 15(m-1) & 0 & 1 \end{pmatrix}.$$

Remark

- The numbers $1, 3, 6, 10, 15, \ldots$ positioned along the $n+2, n$ diagonal of the matrix A correspond to **A000217.**
- The row sums of A for $m = 0$ form the sequence $(1, 1, 0, -2, -4, -4, 0, \ldots)$ corresponds to **A146559** with e.g.f $e^z \cos(z)$.

- The row sums of A for $m=1$ form the sequence $(1,1,1,1,1,1,1,...)$ corresponds to **A000012** with e.g.f e^z.

The production matrix of A in terms of m is given by

$$B=\left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ m-1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2(m-1) & 0 & 1 & 0 & 0 \\ 2\left(m^2-1\right) & 0 & 3(m-1) & 0 & 1 & 0 \\ 0 & 8\left(m^2-1\right) & 0 & 4(m-1) & 0 & 1 \\ 16\left(m^3-2m^2+2m-1\right) & 0 & 20\left(m^2-1\right) & 0 & 5(m-1) & 0 \end{array}\right).$$

If $m=-1,0,1$ then $[\mathrm{cd}(z,m),z]$ produces the Riordan arrays:

$$C=\left\{\left[\mathrm{cd}(z,-1),z\right],\left[\cos(z),z\right],\left[1,z\right]\right\} \text{ respectively.}$$

The production matrices from B if $m=-1,0,1$ are as follows:

$$D=\left\{\left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 & 1 & 0 \\ 0 & 0 & 0 & -8 & 0 & 1 \\ -96 & 0 & 0 & 0 & -10 & 0 \end{array}\right),\left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ -2 & 0 & -3 & 0 & 1 & 0 \\ 0 & -8 & 0 & -4 & 0 & 1 \\ -16 & 0 & -20 & 0 & -5 & 0 \end{array}\right),\left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)\right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D and to the Riordan arrays in C are listed below.

$$\begin{aligned} r(z,m) &= 1\ \forall(m) \\ c(z,m) &= \frac{(m-1)\mathrm{nd}(z|m)\mathrm{sd}(z|m)}{\mathrm{cd}(z|m)} \\ c(z,1) &= 0 \\ c(z,0) &= -\tan(z) \end{aligned}$$

2.3.2.2 $[\mathbf{nd}(z,m),z]$

The case of the exponential Riordan array $[nd(z,m),z]$ has the coefficient matrix given by

$$A=\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ m & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3m & 0 & 1 & 0 & 0 & 0 \\ m(5m-4) & 0 & 6m & 0 & 1 & 0 & 0 \\ 0 & 5m(5m-4) & 0 & 10m & 0 & 1 & 0 \\ m\left(61m^2-76m+16\right) & 0 & 15m(5m-4) & 0 & 15m & 0 & 1 \end{array}\right).$$

Remark

- The numbers $1, 3, 6, 10, 15, ...$ positioned along the $n+2, n$ diagonal of the matrix A correspond to **A000217.**

- The row sums of A for $m = 0$ form the sequence $(1, 1, 1, 1, 1, 1, 1, ...)$ corresponds to **A000012** with e.g.f e^z.

- The row sums of A for $m = 1$ form the sequence $(1, 1, 2, 4, 8, 16, 32, ...)$ corresponds to **A011782** with e.g.f $\cosh(z)e^z = \frac{e^{2z}+1}{2}$.

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ m & 0 & 1 & 0 & 0 & 0 \\ 0 & 2m & 0 & 1 & 0 & 0 \\ 2(m-2)m & 0 & 3m & 0 & 1 & 0 \\ 0 & 8(m-2)m & 0 & 4m & 0 & 1 \\ 16m\left(m^2-m+1\right) & 0 & 20(m-2)m & 0 & 5m & 0 \end{pmatrix}.$$

The production matrices for the case $m = -1, 0, 1$ in B corresponding to the Riordan arrays

$$C = \{[\mathrm{nd}(z|-1), z], [1, z], [\cosh(z), z]\} \text{ respectively,}$$

are given by:

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 6 & 0 & -3 & 0 & 1 & 0 \\ 0 & 24 & 0 & -4 & 0 & 1 \\ -48 & 0 & 60 & 0 & -5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 & 1 & 0 \\ 0 & -8 & 0 & 4 & 0 & 1 \\ 16 & 0 & -20 & 0 & 5 & 0 \end{pmatrix} \right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D are listed as follows:

$$\begin{aligned} r(z, m) &= 1 \; \forall m \\ c(z, m) &= \frac{m\mathrm{cd}(z|m)\mathrm{sd}(z|m)}{\mathrm{nd}(z|m)} \\ c(z, 1) &= \tanh(z) \\ c(z, 0) &= 0 \end{aligned}$$

2.3.2.3 $[\mathbf{nc}(z,m),z]$

The coefficient array of $[nc(z,m),z]$ is given by

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 5-4m & 0 & 6 & 0 & 1 & 0 & 0 \\ 0 & 25-20m & 0 & 10 & 0 & 1 & 0 \\ 16m^2-76m+61 & 0 & 75-60m & 0 & 15 & 0 & 1 \end{pmatrix}.$$

Remark:

- The numbers $1,3,6,10,15,...$ positioned along the $n+2,n$ diagonal of the matrix A correspond to **A000217.**

- The row sums of A for $m=0$ form the sequence $(1,1,2,4,12,36,152,...)$ corresponds to **A003701** with e.g.f $\frac{e^z}{\cos(z)}$.

- The row sums of A for $m=1$ form the sequence $(1,1,2,4,8,16,32,...)$ corresponds to **A011782** with e.g.f $\cosh(z)e^z=\frac{e^{2z}+1}{2}$.

The production matrix of A in terms of m is

$$B=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 2-4m & 0 & 3 & 0 & 1 & 0 \\ 0 & 8-16m & 0 & 4 & 0 & 1 \\ 16\left(m^2-m+1\right) & 0 & 20-40m & 0 & 5 & 0 \end{pmatrix}.$$

The Riordan matrices from $[nc(z,m),z]$ for $m=-1,0,1$ are

$$C=\left\{\left[-1,\mathrm{nc}(z,-1)\right],\left[\sec(z),z\right],\left[\cosh(z),z\right]\right\} \text{ respectively.}$$

The production matrix from B if $m=-1,0,1$ are as follows:

$$D=\left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 6 & 0 & 3 & 0 & 1 & 0 \\ 0 & 24 & 0 & 4 & 0 & 1 \\ 48 & 0 & 60 & 0 & 5 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 8 & 0 & 4 & 0 & 1 \\ 16 & 0 & 20 & 0 & 5 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 & 1 & 0 \\ 0 & -8 & 0 & 4 & 0 & 1 \\ 16 & 0 & -20 & 0 & 5 & 0 \end{pmatrix}\right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D are listed as follows:

$$\begin{aligned} r(z,m) &= 1\ \forall m \\ c(z,m) &= \frac{\text{dn}(z|m)\text{sn}(z|m)}{\text{cn}(z|m)} \\ c(z,1) &= \tanh(z) \\ c(z,0) &= \tan(z). \end{aligned}$$

2.3.3 The Associated subgroup of Jacobi Riordan arrays

Recall from (1.4.1) that the elements of the Associated subgroup are of the form $[1, f(z)]$.

2.3.3.1 $[1, \mathbf{sn}(z,m)]$

The coefficient matrix of $[1, \text{sn}(z,m)]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -m-1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -4(m+1) & 0 & 1 & 0 & 0 \\ 0 & m^2+14m+1 & 0 & -10(m+1) & 0 & 1 & 0 \\ 0 & 0 & 8\left(2m^2+13m+2\right) & 0 & -20(m+1) & 0 & 1 \end{pmatrix}.$$

Remark:

- The columns of matrix A after the first column are **Palindromic.**
- The row sums of A for $m = 0$ form the sequence $(1, 1, 1, 0, -3, -8, -3, ...)$ corresponds to **A002017** with e.g.f $e^{\sin(z)}$.
- The row sums of A for $m = 1$ form the sequence $(1, 1, 1, -1, -7, -3, 97, ...)$ corresponds to **A003723** with e.g.f $e^{\tanh(z)}$.

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -m-1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3(m+1) & 0 & 1 & 0 \\ 0 & -3(m-1)^2 & 0 & -6(m+1) & 0 & 1 \\ 0 & 0 & -15(m-1)^2 & 0 & -10(m+1) & 0 \end{pmatrix}.$$

If $m=-1,0,1$ then $[1,\mathrm{sn}(z,m)]$ produces the Riordan arrays:

$$C=\{[1,\mathrm{sn}(z,-1)],[1,\sin(z)],[1,\tanh(z)]\}.$$

The production matrices from B if $m=-1,0,1$ are as follows:

$$D=\left\{\begin{pmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&-12&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\0&0&1&0&0\\0&-1&0&1&0\\0&0&-3&0&1\\0&-3&0&-6&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0\\0&0&1&0&0\\0&-2&0&1&0\\0&0&-6&0&1\\0&0&0&-12&0\end{pmatrix}\right\}.$$

The generating functions of the r and c sequences corresponding to the production matrix of B and D are listed as follows:

$$\begin{aligned}r(z,m)&=\mathrm{cn}\left(\mathrm{sn}^{-1}(z|m)\middle|\,m\right)\mathrm{dn}\left(\mathrm{sn}^{-1}(z|m)\middle|\,m\right)\\r(z,1)&=1-z^2\\r(z,0)&=\sqrt{1-z^2}\\c(z,m)&=0\ \forall m\end{aligned}$$

2.3.3.2 $\left[1,z^2\mathbf{ns}(z,m)\right]$

The exponential Riordan matrix of $\left[1,z^2ns(z,m)\right]$ is given by

$$A=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&m+1&0&1&0&0&0\\0&0&4(m+1)&0&1&0&0\\0&\frac{1}{3}\left(7m^2-22m+7\right)&0&10(m+1)&0&1&0\\0&0&24\left(m^2-m+1\right)&0&20(m+1)&0&1\end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B=\begin{pmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&m+1&0&1&0&0\\0&0&3(m+1)&0&1&0\\0&\frac{1}{3}\left(-5m^2-46m-5\right)&0&6(m+1)&0&1\\0&0&-\frac{5}{3}\left(5m^2+46m+5\right)&0&10(m+1)&0\end{pmatrix}.$$

The production matrix B for the case $m=-1,0,1$ corresponding to the Riordan arrays

$$C=\left\{\left[1,z^2\mathrm{ns}(z|-1)\right],\left[1,z^2\csc(z)\right],\left[1,z^2\coth(z)\right]\right\},$$

are given by:

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 12 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & -\frac{5}{3} & 0 & 6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 1 \\ 0 & -\frac{56}{3} & 0 & 12 & 0 \end{pmatrix} \right\}.$$

2.3.3.3 $[1, z\mathbf{nc}(z, m)]$

The exponential Riordan matrix of $[1, z\mathrm{nc}(z, m)]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 1 & 0 & 0 \\ 0 & 25-20m & 0 & 30 & 0 & 1 & 0 \\ 0 & 0 & -120(m-2) & 0 & 60 & 0 & 1 \end{pmatrix}.$$

Remark

- The row sums of A for $m = 0$ form the sequence $(1, 1, 1, 4, 13, 56, 301, ...)$ corresponds to **A009300** with e.g.f $e^{\frac{z}{\cos(z)}}$.
- The row sums of A for $m = 1$ form the sequence $(1, 1, 1, 4, 13, 36, 181, ...)$ corresponds to **A003727** with e.g.f $e^{z\cosh(z)}$.

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 & 1 & 0 \\ 0 & -20m-11 & 0 & 18 & 0 & 1 \\ 0 & 0 & -5(20m+11) & 0 & 30 & 0 \end{pmatrix}.$$

The production matrix above for the case $m = 1, 0, -1$ of B corresponding to the Riordan arrays

$$C = \{[1, z\mathrm{nc}(z| -1)], [1, z\sec(z)], [1, z\cosh(z)]\} \text{ respectively}$$

are given by:

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 & 1 \\ 0 & 9 & 0 & 18 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 & 1 \\ 0 & -11 & 0 & 18 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 & 1 \\ 0 & -31 & 0 & 18 & 0 \end{pmatrix} \right\}.$$

2.3.4 Derivative Subgroup of Jacobi Riordan arrays

Recall from (1.4.1) that the elements of the derivative subgroup have the form $(h'(z), h(z))$, where $h'(z)$ denotes the first derivative of $h(z)$.

2.3.4.1 $\left[\frac{d}{dz}\mathbf{sn}(z,m), \mathbf{sn}(z,m)\right]$

The coefficient array of $\left[\frac{d}{dz}\mathrm{sn}(z,m), \mathrm{sn}(z,m)\right]$ where $\frac{d}{dz}\mathrm{sn}(z,m) = \mathrm{cn}(z|m)\mathrm{dn}(z|m)$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -m-1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -4(m+1) & 0 & 1 & 0 & 0 & 0 \\ m^2+14m+1 & 0 & -10(m+1) & 0 & 1 & 0 & 0 \\ 0 & 8\left(2m^2+13m+2\right) & 0 & -20(m+1) & 0 & 1 & 0 \\ -m^3-135m^2-135m-1 & 0 & 91m^2+434m+91 & 0 & -35(m+1) & 0 & 1 \end{pmatrix}$$

Remark:

- The coefficient matrix of $\left[\frac{d}{dz}\mathrm{sn}(z,m), \mathrm{sn}(z,m)\right]$ forms a **Palindromic** Riordan array.

- The row sums of A for $m = 0$ form the sequence $(1, 1, 0, -3, -8, -3, 56, ...)$ which has the e.g.f $\cos(z)e^{\sin(z)}$.

- The row sums of A for $m = 1$ form the sequence $(1, 1, -1, -7, -3, 97, 275, ...)$ which has the e.g.f $\mathrm{sech}^2(z)e^{\tanh(z)}$.

- The non-zero entries of the first column of A corresponds to the matrix

  $$A[1] = \begin{pmatrix} 1 & & & \\ -1 & -1 & & \\ 1 & 14 & 1 & \\ -1 & -135 & -135 & -1 \end{pmatrix}.$$

  By multiplying the matrix $A[1]$ by -1^m if $n \equiv m(\mathrm{mod}2)$ where n is the column number s.t $n = 0, 1, 2, ...$ we get the matrix

  $$B[1] = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 14 & 1 & \\ 1 & 135 & 135 & 1 \end{pmatrix}.$$

  The matrix $B[1]$ corresponds to **A060628.**

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -m-1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3(m+1) & 0 & 1 & 0 & 0 \\ -3(m-1)^2 & 0 & -6(m+1) & 0 & 1 & 0 \\ 0 & -15(m-1)^2 & 0 & -10(m+1) & 0 & 1 \\ -45(m-1)^2(m+1) & 0 & -45(m-1)^2 & 0 & -15(m+1) & 0 \end{pmatrix}$$

If $m = -1, 0, 1$ then $\left[\frac{d}{dz}\mathrm{sn}(z,m), \mathrm{sn}(z,m)\right]$ produces the Riordan arrays

$C = \{[\mathrm{cn}(z,-1)\mathrm{dn}(z,-1), \mathrm{sn}(z,-1)], [\cos(z), \sin(z)], [\mathrm{sech}^2(z), \tanh(z)]\}$ respectively.

The production matrices of B in terms of $m = -1, 0, 1$ are as follows:

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -12 & 0 & 0 & 0 & 1 \\ 0 & -60 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ -3 & 0 & -6 & 0 & 1 \\ 0 & -15 & 0 & -10 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 \\ 0 & 0 & -12 & 0 & 1 \\ 0 & 0 & 0 & -20 & 0 \end{pmatrix} \right\}.$$

REMARK: The tri-diagonal production matrix for $m = 1$ in D which is associated to the Riordan array $[\mathrm{sech}^2(z), \tanh(z)]$ in C forms an orthogonal polynomial sequence for $[\mathrm{sech}^2(z), \tanh(z)]$ $.^{-1}$ Furthermore,

$$[\mathrm{sech}^2(z), \tanh(z)]^{-1} = \left[\frac{1}{1-z^2}, \tanh^{-1}(z)\right]$$

represents the coefficient matrix of the family of formal orthogonal polynomials. The three term recurrence relation for these polynomials is given by

$$P_{n+1}(z) = zP_n(z) + n(n+1)P_{n-1}(z)$$

with $P_0(z) = 1,\ P_1(z) = z$ s.t $-1 < z < 1$.

$$\text{In particular, let } Q_n(z) = \frac{P_n(iz)}{i^n},\ (i^2 = -1)$$

we get

$$Q_{n+1}(z) = zQ_n(z) - n(n+1)Q_{n-1}(z),\ \forall n \geq 1.$$

The generating functions of the r and c sequences corresponding to the production matrix of B and D are listed as follows:

$$\begin{aligned}
\mathrm{A}(z,m) &= \mathrm{cn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right)\mathrm{dn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right)\\
&= \sqrt{(1-\mathrm{sn}^2(\bar{\mathrm{sn}}(z,m),m)(1-m^2\mathrm{sn}^2(\bar{\mathrm{sn}}(z,m),m))}\\
&= \sqrt{(1-z^2)(1-m^2z^2)}\\
\mathrm{A}(z,1) &= 1-z^2\\
\mathrm{A}(z,0) &= \sqrt{1-z^2}\\
\mathrm{Z}(z,m) &= \frac{z\left(m\left(2z^2-1\right)-1\right)\mathrm{cn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right)\mathrm{dn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right)}{\left(z^2-1\right)\left(mz^2-1\right)}\\
&= \frac{z\left(m\left(2z^2-1\right)-1\right)\sqrt{(1-\mathrm{sn}^2(\bar{\mathrm{sn}}(z,m),m)(1-m^2\mathrm{sn}^2(\bar{\mathrm{sn}}(z,m),m))}}{\left(z^2-1\right)\left(mz^2-1\right)}\\
&= \frac{z\left(m\left(2z^2-1\right)-1\right)\sqrt{(1-z^2)(1-m^2z^2)}}{\left(z^2-1\right)\left(mz^2-1\right)}\\
\mathrm{Z}(z,1) &= \frac{-z\left(2z^2-2\right)}{\left(z^2-1\right)}\\
&= -2z\\
\mathrm{Z}(z,0) &= -\frac{z}{\sqrt{1-z^2}}.
\end{aligned}$$

2.3.4.2 $\left[\frac{d}{dz}\mathbf{sc}(z,m), \mathbf{sc}(z,m)\right]$

The coefficient array of $\left[\frac{d}{dz}\mathrm{sc}(z,m), \mathrm{sc}(z,m)\right]$ where $\frac{d}{dz}\mathrm{sc}(z,m) = \mathrm{dc}(z|m)\mathrm{nc}(z|m)$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 2-m & 0 & 1 & 0 & 0\\ 0 & 8-4m & 0 & 1 & 0\\ m^2-16m+16 & 0 & -10(m-2) & 0 & 1 \end{pmatrix}.$$

Remark:

- The row sums of A for $m=0$ form the sequence $(1,1,3,9,37,177,959,...)$ which has the e.g.f $\sec(z)^2 e^{\tan(z)}$.
- The row sums of A for $m=1$ form the sequence $(1,1,2,5,12,37,128,...)$ which has the e.g.f $\cosh(z)e^{\sinh(z)}$.

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 2-m & 0 & 1 & 0 & 0\\ 0 & 6-3m & 0 & 1 & 0\\ -3m^2 & 0 & 12-6m & 0 & 1\\ 0 & -15m^2 & 0 & 20-10m & 0 \end{pmatrix}.$$

If $m=-1,0,1$ then $\left[\frac{d}{dz}\mathrm{sc}(z,m),\mathrm{sc}(z,m)\right]$ produces the Riordan arrays:

$C=\{[\mathrm{dc}(z,-1)\mathrm{nc}(z,-1),\mathrm{sc}(z,-1)],\left[\sec^2(z),\tan(z)\right],[\cosh(z),\sinh(z)]\}$ respectively.

The production matrices from B corresponding to $m=-1,0,1$ are as follows

$$D=\left\{\begin{pmatrix}0&1&0&0\\3&0&1&0\\0&9&0&1\\-3&0&18&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\2&0&1&0\\0&6&0&1\\0&0&12&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&1&0\\0&3&0&1\\-3&0&6&0\end{pmatrix}\right\}.$$

The generating functions of the r and c sequences corresponding to the production matrix of B and D are listed as follows:

$$\begin{aligned}r(z,m)&=\left(z^2+1\right)\mathrm{dn}\left(\mathrm{sc}^{-1}(z|m)\middle|\,m\right)\\r(z,1)&=\sqrt{z^2+1}\\r(z,0)&=z^2+1\\c(z,m)&=\frac{z\left(2(m-1)z^2+m-2\right)\mathrm{dn}\left(\mathrm{sc}^{-1}(z|m)\middle|\,m\right)}{(m-1)z^2-1}\\c(z,1)&=\frac{z}{\sqrt{z^2+1}}\\c(z,0)&=2z.\end{aligned}$$

2.3.4.3 $\left[\frac{d}{dz}\mathbf{am}(z,m),\mathbf{am}(z,m)\right]$

The coefficient array $\left[\frac{d}{dz}\mathrm{am}(z,m),\mathrm{am}(z,m)\right]$ where $\frac{d}{dz}\mathrm{am}(z,m)=\mathrm{dn}(z|m)$ is given by

$$A=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\-m&0&1&0&0&0&0\\0&-4m&0&1&0&0&0\\m(m+4)&0&-10m&0&1&0&0\\0&8m(2m+3)&0&-20m&0&1&0\\-m\left(m^2+44m+16\right)&0&7m(13m+12)&0&-35m&0&1\end{pmatrix}.$$

Remark:

- The row sums of A for $m=0$ form the sequence $(1,1,1,1,1,1,1,...)$ corresponds to **A000012** with e.g.f e^z.

- The row sums of A for $m=1$ form the sequence $(1,1,0,-3,-4,21,80,...)$ corresponds to **A012123** with e.g.f $e^{\sin^{-1}(\tanh(z))}=e^{gdz}$ where g is the

Gudermannian function such that

$$gd(z) = \int_0^z \frac{1}{\cosh t} dt \qquad -\infty < z < \infty.$$

- The non-zero elements of the first column of the matrix A generated from the derivative of the Jacobi amplitude function forms the coefficient matrix

$$A[1] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & -16 & -44 & -1 & 0 & 0 \\ 0 & 64 & 912 & 408 & 1 & 0 \\ 0 & -256 & -15808 & -30768 & -3688 & -1 \end{pmatrix}.$$

The production matrix of A in terms of m:

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -m & 0 & 1 & 0 & 0 & 0 \\ 0 & -3m & 0 & 1 & 0 & 0 \\ (4-3m)m & 0 & -6m & 0 & 1 & 0 \\ 0 & 5(4-3m)m & 0 & -10m & 0 & 1 \\ m\left(-45m^2+60m-16\right) & 0 & 15(4-3m)m & 0 & -15m & 0 \end{pmatrix}.$$

If $m = 1$ and $m = 0$ then $\left[\frac{d}{dz}\text{am}(z,m), \text{am}(z,m)\right]$ produces the Riordan arrays

$$C = \{\left[\frac{2e^z}{e^{2z}+1}, 2\tan^{-1}\left(e^z\right) - \frac{\pi}{2}\right], [1, z]\} \text{ respectively.}$$

Remark: We note that $\frac{2e^z}{e^{2z}+1}$ has an ordinary generating function given by

$$\cfrac{1}{1+\cfrac{x^2}{1+\cfrac{4x^2}{1+\cfrac{9x^2}{1+\cfrac{25x^2}{1+\cdots}}}}}.$$

The production matrices from B in terms of $m = -1, 0, 1$ are as follows

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ -7 & 0 & 6 & 0 & 1 & 0 \\ 0 & -35 & 0 & 10 & 0 & 1 \\ 121 & 0 & -105 & 0 & 15 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 & 0 \\ 1 & 0 & -6 & 0 & 1 & 0 \\ 0 & 5 & 0 & -10 & 0 & 1 \\ -1 & 0 & 15 & 0 & -15 & 0 \end{pmatrix} \right\}.$$

Remark: We note that the generating function of the matrix D at $m = 1$ is

$$e^{xy}(-\sin(x) + y\cos(x)).$$

The generating functions of the r and c sequences corresponding to the production matrix of B and D are listed as follows:

$$\begin{aligned}
r(z,m) &= \mathrm{dn}(F(z|m)|m) \\
r(z,1) &= \mathrm{sech}(F(z|1)) \\
&= \mathrm{sech}(\log(\tan(z)+\sec(z))) \\
&= \frac{2(\tan(z)+\sec(z))}{(\tan(z)+\sec(z))^2+1} \\
&= \cos(z) \\
r(z,0) &= 1 \\
c(z,m) &= \frac{m\mathrm{cn}(F(z|m)|m)\mathrm{sn}(F(z|m)|m)}{\mathrm{dn}(F(z|m)|m)} \\
c(z,1) &= -\tanh(F(z|1)) \\
&= -\tanh(\log(\tan(z)+\sec(z))) \\
&= \frac{1-(\tan(z)+\sec(z))^2}{1+(\tan(z)+\sec(z))^2} \\
&= \sin(z) \\
c(z,0) &= 0
\end{aligned}$$

Recall: The incomplete elliptic integral of the first kind (2.1) denoted $F(\phi, m)$ where $0 < \phi < \frac{\pi}{2}$. If $\phi = \frac{\pi}{2}$ we have the complete elliptic integral of the first kind K. It has a series expansion given by

$$F(\phi,m) = \phi + \frac{m\phi^3}{6} + \frac{1}{120}\left(9m^2 - 4m\right)\phi^5 + \frac{\left(225m^3 - 180m^2 + 16m\right)\phi^7}{5040} + O\left(\phi^9\right).$$

The *Mathematica* code to compute $F(z|1)$ is as follows: **EllipticF[z, 1]**

EllipticF[z, 1]

FunctionExpand [%, 0 < z < $\frac{\mathbf{Pi}}{\mathbf{2}}$]

Log[Sec[z] + Tan[z]]

2.3.4.4 $\left[\frac{d}{dz}\mathbf{sd}(z,m),\mathbf{sd}(z,m)\right]$

The coefficient array of $\left[\frac{d}{dz}\mathrm{sd}(z,m),\mathrm{sd}(z,m)\right]$ where $\frac{d}{dz}\mathrm{sd}(z,m)=\mathrm{cd}(z|m)\mathrm{nd}(z|m)$ is given by

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 2\left(m-\frac{1}{2}\right) & 0 & 1 & 0 & 0 & 0\\ 0 & 6\left(\frac{4m}{3}-\frac{2}{3}\right) & 0 & 1 & 0 & 0\\ 16m^2-16m+1 & 0 & 12\left(\frac{5m}{3}-\frac{5}{6}\right) & 0 & 1 & 0\\ 0 & 8\left(17m^2-17m+2\right) & 0 & 20(2m-1) & 0 & 1 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 2m-1 & 0 & 1 & 0 & 0 & 0\\ 0 & 8m-4 & 0 & 1 & 0 & 0\\ 16m^2-16m+1 & 0 & 20m-10 & 0 & 1 & 0\\ 0 & 8\left(17m^2-17m+2\right) & 0 & 40m-20 & 0 & 1 \end{pmatrix}.$$

REMARK:

- The numbers $2,6,12,20,30,42,...$ located on the $n+2,n$ diagonal correspond to **A002378**.
- The row sums of A for $m=0$ form the sequence $(1,1,0,-3,-8,-3,56,...)$ with e.g.f $\cos(z)e^{\sin(z)}$.
- The row sums of A for $m=1$ form the sequence $(1,1,2,5,12,37,128,...)$ which has e.g.f $\cosh(z)e^{\sinh(z)}$.

If $m=1$ and $m=0$ then $\left[\frac{d}{dz}\mathrm{sd}(z,m),\mathrm{sd}(z,m)\right]$ produces the Riordan arrays $C=\{[\cosh(z),\sinh(z)],[\cos(z),\sin(z)]\}$. The production matrix of A in terms of m is given by

$$B=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 2m-1 & 0 & 1 & 0 & 0 & 0\\ 0 & 6m-3 & 0 & 1 & 0 & 0\\ -3 & 0 & 12m-6 & 0 & 1 & 0\\ 0 & -15 & 0 & 20m-10 & 0 & 1\\ 90m-45 & 0 & -45 & 0 & 30m-15 & 0 \end{pmatrix}.$$

The production matrices from B in terms of $m=-1,0,1$ are as follows:

$$D=\left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ -3 & 0 & 1 & 0 & 0\\ 0 & -9 & 0 & 1 & 0\\ -3 & 0 & -18 & 0 & 1\\ 0 & -15 & 0 & -30 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ -1 & 0 & 1 & 0 & 0\\ 0 & -3 & 0 & 1 & 0\\ -3 & 0 & -6 & 0 & 1\\ 0 & -15 & 0 & -10 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0\\ 0 & 3 & 0 & 1 & 0\\ -3 & 0 & 6 & 0 & 1\\ 0 & -15 & 0 & 10 & 0 \end{pmatrix}\right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D are listed as follows:

$$\begin{aligned}
r(z,m) &= \operatorname{cd}\left(\operatorname{sd}^{-1}(z|m)\middle|\, m\right)\operatorname{nd}\left(\operatorname{sd}^{-1}(z|m)\middle|\, m\right)\\
r(z,1) &= \sqrt{z^2+1}\\
r(z,0) &= \sqrt{1-z^2}\\
c(z,m) &= \frac{z\left(2(m-1)mz^2+2m-1\right)\operatorname{cn}\left(\operatorname{sd}^{-1}(z|m)\middle|\, m\right)}{(m-1)z^2+1}\\
c(z,1) &= \frac{z}{\sqrt{z^2+1}}\\
c(z,0) &= -\frac{z}{\sqrt{1-z^2}}.
\end{aligned}$$

2.3.4.5 $\left[\frac{d}{dz}\mathbf{sc}^{-1}(z,m), \mathbf{sc}^{-1}(z,m)\right]$

The coefficient matrix of $\left[\frac{d}{dz}\operatorname{sc}^{-1}(z,m), \operatorname{sc}^{-1}(z,m)\right]$ where $\frac{d}{dz}\operatorname{sc}^{-1}(z,m) = \frac{\operatorname{nd}\left(\operatorname{sc}^{-1}(z|m)\middle| m\right)}{z^2+1}$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ m-2 & 0 & 1 & 0 & 0\\ 0 & 4(m-2) & 0 & 1 & 0\\ 3\left(3m^2-8m+8\right) & 0 & 10(m-2) & 0 & 1 \end{pmatrix}.$$

Remark: The row sums of A for $m=0$ form the sequence $(1,1,-1,-7,5,145,-5,...)$ which has the e.g.f $\frac{1}{1+z^2}e^{\tan^{-1}(z)}$. The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ m-2 & 0 & 1 & 0 & 0\\ 0 & 3(m-2) & 0 & 1 & 0\\ m(5m-8)+8 & 0 & 6(m-2) & 0 & 1\\ 0 & 5(m(5m-8)+8) & 0 & 10(m-2) & 0 \end{pmatrix}.$$

If $m=1$ and $m=0$ then $\left[\frac{d}{dz}\operatorname{sc}^{-1}(z,m), \operatorname{sc}^{-1}(z,m)\right]$ produces the Riordan arrays $C = \left\{\left[\frac{1}{\sqrt{z^2+1}}, \sinh^{-1}(z)\right], \left[\frac{1}{z^2+1}, \tan^{-1}(z)\right]\right\}$. The production matrices from B for $m=-1,0,1$ are as follows:

$$D = \left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ -3 & 0 & 1 & 0 & 0\\ 0 & -9 & 0 & 1 & 0\\ 21 & 0 & -18 & 0 & 1\\ 0 & 105 & 0 & -30 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ -2 & 0 & 1 & 0 & 0\\ 0 & -6 & 0 & 1 & 0\\ 8 & 0 & -12 & 0 & 1\\ 0 & 40 & 0 & -20 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ -1 & 0 & 1 & 0 & 0\\ 0 & -3 & 0 & 1 & 0\\ 5 & 0 & -6 & 0 & 1\\ 0 & 25 & 0 & -10 & 0 \end{pmatrix}\right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D are listed as follows:

$$\begin{aligned} r(z,m) &= \operatorname{cn}(z|m)^2 \operatorname{nd}\left(\operatorname{sc}^{-1}(\operatorname{sc}(z|m)|m)\middle|\, m\right) \\ r(z,1) &= \frac{1}{\cosh(z)} \\ r(z,0) &= \cos^2(z) \\ c(z,1) &= \tanh(z)(-\operatorname{sech}(z)) \\ c(z,0) &= -\sin(2z) \end{aligned}$$

$$c(z,m) = \operatorname{cn}(z|m)^4 \left(\operatorname{sc}(z|m)^2+1\right)\left(m\operatorname{cd}\left(\operatorname{sc}^{-1}(\operatorname{sc}(z|m)|m)\middle|\, m\right)\operatorname{sd}\left(\operatorname{sc}^{-1}(\operatorname{sc}(z|m)|m)\middle|\, m\right) - 2\operatorname{sc}(z|m)\right).$$

2.3.4.6 $\left[\frac{d}{dz}\mathbf{sd}^{-1}(z|m), \mathbf{sd}^{-1}(z|m)\right]$

The coefficient matrix $\left[\frac{d}{dz}\operatorname{sd}^{-1}(z|m), \operatorname{sd}^{-1}(z|m)\right]$ where

$$\frac{d}{dz}\operatorname{sd}^{-1}(z|m) = \frac{\operatorname{cn}\left(\operatorname{sd}^{-1}(z|m)\middle|\, m\right)}{(m-1)z^2+1}.$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2\left(\frac{1}{2}-m\right) & 0 & 1 & 0 & 0 \\ 0 & 6\left(\frac{2}{3}-\frac{4m}{3}\right) & 0 & 1 & 0 \\ 24\left(m^2-m+\frac{3}{8}\right) & 0 & 12\left(\frac{5}{6}-\frac{5m}{3}\right) & 0 & 1 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1-2m & 0 & 1 & 0 & 0 \\ 0 & 4-8m & 0 & 1 & 0 \\ 3\left(8m^2-8m+3\right) & 0 & 10-20m & 0 & 1 \end{pmatrix}.$$

REMARK:

- The numbers $2, 6, 12, 20, 30, 42, ...$ located on the $n+2, n$ diagonal correspond to **A002378**
- The row sums of A for $m = 0$ form the sequence $(1, 1, 2, 5, 20, 85, 520, ...)$ which has e.g.f $\frac{1}{\sqrt{1-z^2}} e^{\sin^{-1}(z)}$.

The production matrix of A in terms of m:

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-2m & 0 & 1 & 0 \\ 0 & 3-6m & 0 & 1 \\ 8m^2-8m+5 & 0 & 6-12m & 0 \end{pmatrix}.$$

If $m = 1$ and $m = 0$ then $\left[\frac{d}{dz}\text{sd}^{-1}(z,m), \text{sd}^{-1}(z,m)\right]$ produces the Riordan arrays $C = \{\left[\frac{1}{\sqrt{z^2+1}}, \sinh^{-1}(z)\right], \left[\frac{1}{\sqrt{1-z^2}}, \sin^{-1}(z)\right]\}$ respectively. In particular we have that,

$$\left[\frac{1}{\sqrt{z^2+1}}, \sinh^{-1}(z)\right]^{-1} = \left[\cosh(z), \sinh(z)\right].$$

The production matrices from B for $m = -1, 0, 1$ are as follows

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \\ 21 & 0 & 18 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 5 & 0 & 6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 5 & 0 & -6 & 0 \end{pmatrix} \right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D are listed as follows:

$$\begin{aligned} r(z,m) &= \frac{\text{cn}\left(\text{sd}^{-1}(\text{sd}(z|m)|m)\middle| m\right)}{(m-1)\text{sd}(z|m)^2+1} \\ r(z,1) &= \frac{1}{\cosh(z)} \\ r(z,0) &= \frac{1}{\cos(z)} \\ c(z,m) &= -\frac{\text{dn}\left(\text{sd}^{-1}(\text{sd}(z|m)|m)\middle| m\right)\text{sn}\left(\text{sd}^{-1}(\text{sd}(z|m)|m)\middle| m\right)+2(m-1)\text{sd}(z|m)}{(m-1)\text{sd}(z|m)^2+1} \\ c(z,1) &= \tanh(z)(-\text{sech}(z)) \\ c(z,0) &= \tan(z)\sec(z) \end{aligned}$$

2.3.4.7 $\left[\frac{d}{dz}\mathbf{sn}^{-1}(z,m), \mathbf{sn}^{-1}(z,m)\right]$

The coefficient array of $\left[\frac{d}{dz}\text{sn}^{-1}(z,m), \text{sn}^{-1}(z,m)\right]$ where

$$\frac{d}{dz}\text{sn}^{-1}(z,m) = \frac{\text{cd}\left(\text{sn}^{-1}(z,m)\middle| m\right)}{1-z^2}$$

is given by

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ m+1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 4(m+1) & 0 & 1 & 0 & 0 \\ 3\left(3m^2+2m+3\right) & 0 & 10(m+1) & 0 & 1 & 0 \\ 0 & 8\left(8m^2+7m+8\right) & 0 & 20(m+1) & 0 & 1 \end{pmatrix}.$$

REMARK:

- The numbers $1, 4, 10, 20, \ldots$ located on the $(n+2, n)$ diagonal correspond to **A000292.**
- The row sums of A for $m=0$ form the sequence $(1, 1, 2, 5, 20, 85, 520, \ldots)$ which has e.g.f $\frac{1}{\sqrt{1-z^2}} e^{\sin^{-1}(z)}$.
- The row sums of A for $m=1$ form the sequence $(1, 1, 3, 9, 45, 225, 1575, \ldots)$ which has e.g.f $\frac{1}{1-z^2}\sqrt{\frac{1+z}{1-z}}$.

The production matrix of A expressed in terms of m

$$B=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ m+1 & 0 & 1 & 0 & 0 \\ 0 & 3(m+1) & 0 & 1 & 0 \\ 5m^2-2m+5 & 0 & 6(m+1) & 0 & 1 \\ 0 & 5\left(5m^2-2m+5\right) & 0 & 10(m+1) & 0 \end{pmatrix}.$$

If $m=1$ and $m=0$ then $\left[\frac{d}{dz}\operatorname{sn}^{-1}(z,m), \operatorname{sn}^{-1}(z,m)\right]$ produces the Riordan arrays $C=\left[\frac{1}{1-z^2}, \tanh^{-1}(z)\right], \left[\frac{1}{\sqrt{1-z^2}}, \sin^{-1}(z)\right]$ respectively. In particular,

$$\left[\frac{1}{1-z^2}, \tanh^{-1}(z)\right]^{-1} = \left[\operatorname{sech}^2(\mathrm{z}), \tanh(z)\right]$$

corresponds to a signed version of **A059419**. We recall that $\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$. The production matrices from B in terms of $m=-1,0,1$ are as follows:

$$D=\left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 12 & 0 & 0 & 0 & 1 \\ 0 & 60 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 5 & 0 & 6 & 0 & 1 \\ 0 & 25 & 0 & 10 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & 1 & 0 \\ 8 & 0 & 12 & 0 & 1 \\ 0 & 40 & 0 & 20 & 0 \end{pmatrix}\right\}.$$

The generating functions of the r and c sequence corresponding to the production matrices of B and D are listed as follows:

$$\begin{aligned}
r(z,m) &= \frac{\mathrm{cd}\left(\mathrm{sn}^{-1}(\mathrm{sn}(z|m)|m)\middle|\, m\right)}{\mathrm{cn}(z|m)^2}\\
r(z,1) &= \cosh^2(z)\\
r(z,0) &= \frac{1}{\sqrt{\cos^2(z)}}\\
c(z,m) &= \frac{\left(\frac{m-1}{\mathrm{dn}(z|m)^2}+2\right)\mathrm{sn}(z|m)}{\mathrm{cn}(z|m)^2}\\
c(z,1) &= \sinh(2z)\\
c(z,0) &= \tan(z)\sec(z).
\end{aligned}$$

2.4 Products of Jacobi Riordan arrays

We use the multiplication rule of Riordan arrays (1.3) in the context of Riordan arrays defined by Jacobi elliptic functions.

2.4.1 $[\mathbf{cn}(z,m), \mathbf{sn}(z,m)]^2$

Consider the product $[\mathrm{cn}(z,m), \mathrm{sn}(z,m)]^2 = [\mathrm{cn}(z,m), \mathrm{sn}(z,m)]\,[\mathrm{cn}(z,m), \mathrm{sn}(z,m)] \equiv [\mathrm{cn}(z,m)\mathrm{cn(sn(z,m),m)}, \mathrm{sn}(sn(z,m),m)]$. The resulting coefficient matrix is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ -2 & 0 & 1 & 0 & 0\\ 0 & 6\left(-\frac{m}{3}-\frac{4}{3}\right) & 0 & 1 & 0\\ 12(m+1) & 0 & 12\left(-\frac{2m}{3}-\frac{5}{3}\right) & 0 & 1 \end{pmatrix}.$$

which is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ -2 & 0 & 1 & 0 & 0\\ 0 & -2(m+4) & 0 & 1 & 0\\ 12(m+1) & 0 & -4(2m+5) & 0 & 1 \end{pmatrix}.$$

REMARK The numbers $2, 6, 12, 20, 30, ...$ located on the $(n+2, n)$ diagonal of A correspond to **A002378.**

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -2(m+3) & 0 & 1 \\ 8m-4 & 0 & -6(m+2) & 0 \end{pmatrix}.$$

If $m = 1$ and $m = 0$ then $[\mathrm{cn}(z,m), \mathrm{sn}(z,m)]^2$ produces the Riordan arrays $C = \{[\mathrm{sech}(z)\mathrm{sech}(\tanh(z)), \tanh(\tanh(z))], [\cos(z)\cos(\sin(z)), \sin(\sin(z))]\}$ respectively. The production matrices from B in terms of $m = -1, 0, 1$ are as follows:

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 \\ -12 & 0 & -6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -6 & 0 & 1 \\ -4 & 0 & -12 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -8 & 0 & 1 \\ 4 & 0 & -18 & 0 \end{pmatrix} \right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D are listed as follows:

$$r(z,m) = \mathrm{cn}\left(\mathrm{sn}^{-1}(z|m)\middle| m\right) \mathrm{cn}\left(\mathrm{sn}^{-1}\left(\mathrm{sn}^{-1}(z|m)\middle| m\right)\middle| m\right) \times \\ \mathrm{dn}\left(\mathrm{sn}^{-1}(z|m)\middle| m\right) \mathrm{dn}\left(\mathrm{sn}^{-1}\left(\mathrm{sn}^{-1}(z|m)\middle| m\right)\middle| m\right)$$

$$\begin{aligned} r(z,1) &= \left(z^2-1\right)\left(\tanh^{-1}(z)^2-1\right) \\ r(z,0) &= \sqrt{1-z^2}\sqrt{1-\sin^{-1}(z)^2} \\ c(z,1) &= \tanh^{-1}(z)\left(z\tanh^{-1}(z)-1\right)-z \\ c(z,0) &= \frac{\sin^{-1}(z)\left(\sqrt{1-z^2}-z\sin^{-1}(z)\right)+z}{\sqrt{1-z^2}\sqrt{1-\sin^{-1}(z)^2}} \end{aligned}$$

2.4.2 $[\mathbf{cn}(z,m), \mathbf{sn}(z,m)]\left[\frac{d}{dz}\mathbf{sn}(z,m), \mathbf{sn}(z,m)\right]$

Consider the product

$$[\mathrm{cn}(z,m), \mathrm{sn}(z,m)]\left[\frac{d}{dz}\mathrm{sn}(z,m), \mathrm{sn}(z,m)\right] = [\mathrm{cn}(z,m), \mathrm{sn}(z,m)][\mathrm{cn}(z,m)\mathrm{dn}(z,m), \mathrm{sn}(z,m)]$$

$$= [\mathrm{cn}(z,m)\mathrm{cn}(sn(z,m),m)\mathrm{dn}(sn(z,m),m), \mathrm{sn}(sn(z,m),m)].$$

The resulting coefficient matrix is given by :

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2\left(-\frac{m}{2}-1\right) & 0 & 1 & 0 & 0 \\ 0 & 6\left(-\frac{5m}{6}-\frac{4}{3}\right) & 0 & 1 & 0 \\ 5m^2+32m+12 & 0 & 12\left(-\frac{7m}{6}-\frac{5}{3}\right) & 0 & 1 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -m-2 & 0 & 1 & 0 & 0 \\ 0 & -5m-8 & 0 & 1 & 0 \\ 5m^2+32m+12 & 0 & -2(7m+10) & 0 & 1 \end{pmatrix}.$$

REMARK The numbers $2, 6, 12, 20, 30, ...$ located on the $n+2, n$ diagonal of A correspond to **A002378.**

The production matrix of A in terms of m : is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -m-2 & 0 & 1 & 0 \\ 0 & -4m-6 & 0 & 1 \\ 14m-4 & 0 & -9m-12 & 0 \end{pmatrix}.$$

The production matrices from B in terms of $m = -1, 0, 1$: are as follows

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ -18 & 0 & -3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -6 & 0 & 1 \\ -4 & 0 & -12 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & -10 & 0 & 1 \\ 10 & 0 & -21 & 0 \end{pmatrix} \right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D are listed as follows:

$$r(z,m) = \operatorname{cn}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right) \operatorname{cn}\left(\operatorname{sn}^{-1}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right)\middle| m\right) \operatorname{dn}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right) \times \operatorname{dn}\left(\operatorname{sn}^{-1}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right)\middle| m\right)$$

$$r(z,1) = \left(z^2-1\right)\left(\tanh^{-1}(z)^2-1\right)$$

$$r(z,0) = \sqrt{1-z^2}\sqrt{1-\sin^{-1}(z)^2}$$

$$c(z,1) = -\sqrt{1-z^2}\left(z\tanh^{-1}(z)-1\right)\left(\tanh^{-1}(z)^2-1\right)\tanh\left(\sqrt{1-z^2}\tanh^{-1}(z)\right) - \tanh^{-1}(z)$$

$$c(z,0) = -\frac{\sin^{-1}(z)\left(\sqrt{1-z^2}-z\sin^{-1}(z)\right)+z}{\sqrt{1-z^2}\sqrt{1-\sin^{-1}(z)^2}}$$

2.4.3 $\left[\frac{d}{dz}\mathbf{sn}(z,m), \mathbf{sn}(z,m)\right]^2$

Consider the product:

$$\left[\frac{d}{dz}\text{sn}(z,m), \text{sn}(z,m)\right]^2 = \left[\frac{d}{dz}\text{sn}(z,m), \text{sn}(z,m)\right]\left[\frac{d}{dz}\text{sn}(z,m), \text{sn}(z,m)\right]$$

$$= [\text{cn(z,m)dn(z,m)}, \text{sn}(z,m)]\,[\text{cn(z,m)dn(z,m)}, \text{sn}(z,m)]$$

$$= [\text{cn(z,m)dn(z,m)cn(sn(z,m),m)dn(sn(z,m),m)}, \text{sn}(sn(z,m),m)]\,.$$

The resulting coefficient matrix is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2(-m-1) & 0 & 1 & 0 & 0 \\ 0 & -8(m+1) & 0 & 1 & 0 \\ 12\left(m^2+4m+1\right) & 0 & -20(m+1) & 0 & 1 \end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2(m+1) & 0 & 1 & 0 \\ 0 & -6(m+1) & 0 & 1 \\ -4\left(m^2-4m+1\right) & 0 & -12(m+1) & 0 \end{pmatrix}.$$

If $m=1$ and $m=0$ then $\left[\frac{d}{dz}\text{sn}(z,m), \text{sn}(z,m)\right]^2$ produces the Riordan arrays $\left\{\left[\text{sech}^2(z)\text{sech}^2(\tanh(z)), \tanh(\tanh(z))\right], [\cos(z)\cos(\sin(z)), \sin(\sin(z))]\right\}$ respectively.

The production matrices from B in terms of $m=-1,0,1$ are as follows:

$$D = \left\{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -24 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -6 & 0 & 1 \\ -4 & 0 & -12 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & -12 & 0 & 1 \\ 8 & 0 & -24 & 0 \end{pmatrix}\right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D are listed as follows:

$$r(z,m) = \operatorname{cn}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right) \operatorname{cn}\left(\operatorname{sn}^{-1}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right)\middle| m\right) \operatorname{dn}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right) \times \\ \operatorname{dn}\left(\operatorname{sn}^{-1}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right)\middle| m\right)$$

$$r(z,1) = \left(z^2-1\right)\left(\tanh^{-1}(z)^2-1\right)$$

$$r(z,0) = \sqrt{1-z^2}\sqrt{1-\sin^{-1}(z)^2}$$

$$c(z,1) = 2\tanh^{-1}(z)\left(z\tanh^{-1}(z)-1\right)-2z$$

$$c(z,0) = -\frac{\sin^{-1}(z)\left(\sqrt{1-z^2}-z\sin^{-1}(z)\right)+z}{\sqrt{1-z^2}\sqrt{1-\sin^{-1}(z)^2}}$$

2.4.4 $\left[\frac{d}{dz}\mathbf{sc}(z,m), \mathbf{sc}(z,m)\right]\left[\frac{d}{dz}\mathbf{sn}(z,m), \mathbf{sn}(z,m)\right]$

Consider the product

$$\left[\frac{d}{dz}\operatorname{sc}(z,m), \operatorname{sc}(z,m)\right]\left[\frac{d}{dz}\operatorname{sn}(z,m), \operatorname{sn}(z,m)\right]$$
$$= [\text{dc(z,m)nc(z,m)}, \operatorname{sn}(z,m)]\,[\text{cn(z,m)dn(z,m)}, \operatorname{sn}(z,m)]$$
$$= [\text{dc(z,m)nc(z,m)cn(sc(z,m),m)dn(sc(z,m),m)}, \operatorname{sn}(sc(z,m),m)]\,.$$

The resulting coefficient matrix is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2\left(\frac{1}{2}-m\right) & 0 & 1 & 0 & 0 \\ 0 & 6\left(\frac{2}{3}-\frac{4m}{3}\right) & 0 & 1 & 0 \\ 3\left(4m^2-4m-1\right) & 0 & 12\left(\frac{5}{6}-\frac{5m}{3}\right) & 0 & 1 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1-2m & 0 & 1 & 0 & 0 \\ 0 & 4-8m & 0 & 1 & 0 \\ 3\left(4m^2-4m-1\right) & 0 & 10-20m & 0 & 1 \end{pmatrix}.$$

REMARK The numbers $2, 6, 12, 20, 30, \ldots$ located on the $n+2, n$ diagonal correspond to **A002378.**

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-2m & 0 & 1 & 0 \\ 0 & 3-6m & 0 & 1 \\ -4m^2+4m-7 & 0 & 6-12m & 0 \end{pmatrix}.$$

The production matrices from B in terms of $m = -1, 0, 1$ are as follows:

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \\ -15 & 0 & 18 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ -7 & 0 & 6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ -7 & 0 & -6 & 0 \end{pmatrix} \right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D are listed as follows:

$$r(z,m) = \left(\operatorname{sn}^{-1}(z|m)^2+1\right) \operatorname{cn}\left(\operatorname{sn}^{-1}(z|m)\middle|\, m\right) \operatorname{dn}\left(\operatorname{sn}^{-1}(z|m)\middle|\, m\right) \times \operatorname{dn}\left(\operatorname{sc}^{-1}\left(\operatorname{sn}^{-1}(z|m)\middle|\, m\right)\middle|\, m\right)$$

$$r(z,1) = \left(1-z^2\right)\sqrt{\tanh^{-1}(z)^2+1}$$

$$r(z,0) = \sqrt{1-z^2}\left(\sin^{-1}(z)^2+1\right)$$

$$c(z,m) =$$

$$c(z,1) = \frac{-2z - 2z\tanh^{-1}(z)^2 + \tanh^{-1}(z)}{\sqrt{\tanh^{-1}(z)^2+1}}$$

$$c(z,0) = \frac{2\sqrt{1-z^2}\sin^{-1}(z) - z\left(\sin^{-1}(z)^2+1\right)}{\sqrt{1-z^2}}$$

2.4.5 $\left[\frac{d}{dz}\mathbf{sd}(z,m), \mathbf{sd}(z,m)\right]\left[\frac{d}{dz}\mathbf{sn}(z,m), \mathbf{sn}(z,m)\right]$

Consider the product

$$\left[\frac{d}{dz}\operatorname{sd}(z,m), \operatorname{sd}(z,m)\right]\left[\frac{d}{dz}\operatorname{sn}(z,m), \operatorname{sn}(z,m)\right]$$

$$= [\text{cd(z,m)nd(z,m)}, \operatorname{sd}(z,m)]\,[\text{cn(z,m)dn(z,m)}, \operatorname{sn}(z,m)]$$

$$= [\text{cd(z,m)nd(z,m)cn(sd(z,m),m)dn(sd(z,m),m)}, \operatorname{sd}(sd(z,m),m)]\,.$$

The resulting coefficient array is given by :

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ m-2 & 0 & 1 & 0 & 0 \\ 0 & 4(m-2) & 0 & 1 & 0 \\ -3\left(m^2+4m-4\right) & 0 & 10(m-2) & 0 & 1 \end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ m-2 & 0 & 1 & 0 \\ 0 & 3(m-2) & 0 & 1 \\ -7m^2+4m-4 & 0 & 6(m-2) & 0 \end{pmatrix}. .$$

The production matrices from B in terms of $m = \{-1, 0, 1\}$ are as follows:

$$D = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & -9 & 0 & 1 \\ -15 & 0 & -18 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -6 & 0 & 1 \\ -4 & 0 & -12 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ -7 & 0 & -6 & 0 \end{pmatrix} \right\}.$$

The generating functions of the r and c sequences corresponding to the production matrices of B and D are listed as follows:

$$r(z,m) = \operatorname{cn}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right) \operatorname{dn}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right) \operatorname{cd}\left(\operatorname{sd}^{-1}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right)\middle| m\right) \times \operatorname{nd}\left(\operatorname{sd}^{-1}\left(\operatorname{sn}^{-1}(z|m)\middle| m\right)\middle| m\right)$$

$$r(z,1) = \left(1-z^2\right)\sqrt{\tanh^{-1}(z)^2+1}$$

$$r(z,0) = \sqrt{1-z^2}\sqrt{1-\sin^{-1}(z)^2}$$

$$c(z,1) = \frac{-2z-2z\tanh^{-1}(z)^2+\tanh^{-1}(z)}{\sqrt{\tanh^{-1}(z)^2+1}}$$

$$c(z,0) = -\frac{\sqrt{1-z^2}\sin^{-1}(z)+z-z\sin^{-1}(z)^2}{\sqrt{1-z^2}\sqrt{1-\sin^{-1}(z)^2}}$$

2.4.6 $\left[\frac{d}{dz}\mathbf{sc}^{-1}(z,m),\mathbf{sc}^{-1}(z,m)\right]\left[\frac{d}{dz}\mathbf{sn}(z,m),\mathbf{sn}(z,m)\right]$

For the case of $\left[\frac{d}{dz}\mathrm{sc}^{-1}(z,m),\mathrm{sc}^{-1}(z,m)\right]\left[\frac{d}{dz}\mathrm{sn}(z,m),\mathrm{sn}(z,m)\right]$ its coefficient array is given by

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3(m+3) & 0 & 1 & 0 & 0 & 0 \\ 45 & 0 & -6(2m+3) & 0 & 1 & 0 & 0 \\ 0 & 45\left(m^2+2m+5\right) & 0 & -30(m+1) & 0 & 1 & 0 \\ -1575 & 0 & 45\left(8m^2+12m+15\right) & 0 & -15(4m+3) & 0 & 1 \end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3(m+2) & 0 & 1 & 0 & 0 \\ -9(m-2) & 0 & -9(m+1) & 0 & 1 & 0 \\ 0 & 9\left(m^2-4m+8\right) & 0 & -6(3m+2) & 0 & 1 \\ -45\left(3m^2-12m+8\right) & 0 & 45\left(m^2-2m+4\right) & 0 & -15(2m+1) & 0 \end{pmatrix}.$$

The production matrices of B interms of $m=-1,0,1$ are respectively given by

$$\left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 & 0 \\ 27 & 0 & 0 & 0 & 1 & 0 \\ 0 & 117 & 0 & 6 & 0 & 1 \\ -1035 & 0 & 315 & 0 & 15 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 & 0 \\ 18 & 0 & -9 & 0 & 1 & 0 \\ 0 & 72 & 0 & -12 & 0 & 1 \\ -360 & 0 & 180 & 0 & -15 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & -9 & 0 & 1 & 0 & 0 \\ 9 & 0 & -18 & 0 & 1 & 0 \\ 0 & 45 & 0 & -30 & 0 & 1 \\ 45 & 0 & 135 & 0 & -45 & 0 \end{pmatrix}\right\}.$$

2.5 Jacobi Riordan arrays of complex variables with imaginary modulus k'

In this section we construct Riordan arrayus using Jacobi elliptic functions of a complex variable u with an imaginary modulus k'.

2.5.1 The Riordan array from $\mathbf{sn}(iu,k')$

The imaginary transformation of the elliptic sinus is $\mathrm{sn}(iu,k')=i\frac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)}$ having series expansion given by

$$i\frac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)}=iu-\frac{1}{6}i(k-2)u^3+\frac{1}{120}i\left(k^2-16k+16\right)u^5-\frac{i\left(k^3-138k^2+408k-272\right)u^7}{5040}+O\left(u^9\right). \quad (2.3)$$

2.5.1.1 $\left[1, i\frac{\mathbf{sn}(u,k)}{\mathbf{cn}(u,k)}\right]$

The exponential Riordan array of the Lagrange subgroup corresponding to (2.3) is $\left[1, i\frac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)}\right]$ having coefficient matrix given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -i(k-2) & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 4(k-2) & 0 & 1 & 0 & 0 \\ 0 & i\left(k^2-16k+16\right) & 0 & 10i(k-2) & 0 & i & 0 \\ 0 & 0 & 8\left(-2k^2+17k-17\right) & 0 & -20(k-2) & 0 & -1 \end{pmatrix}.$$

The production matrix of A in terms of k is given by

$$B = \begin{pmatrix} 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & i(k-2) & 0 & i & 0 & 0 \\ 0 & 0 & 3i(k-2) & 0 & i & 0 \\ 0 & -3ik^2 & 0 & 6i(k-2) & 0 & i \\ 0 & 0 & -15ik^2 & 0 & 10i(k-2) & 0 \end{pmatrix}.$$

The production matrix of B for the cases $k = -1, 0, 1$ is given by

$$\left\{ \begin{pmatrix} 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & -3i & 0 & i & 0 & 0 \\ 0 & 0 & -9i & 0 & i & 0 \\ 0 & -3i & 0 & -18i & 0 & i \\ 0 & 0 & -15i & 0 & -30i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & -2i & 0 & i & 0 & 0 \\ 0 & 0 & -6i & 0 & i & 0 \\ 0 & 0 & 0 & -12i & 0 & i \\ 0 & 0 & 0 & 0 & -20i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & -i & 0 & i & 0 & 0 \\ 0 & 0 & -3i & 0 & i & 0 \\ 0 & -3i & 0 & -6i & 0 & i \\ 0 & 0 & -15i & 0 & -10i & 0 \end{pmatrix} \right\}.$$

2.5.1.2 $\left[\frac{d}{du}\left(i\frac{\mathbf{sn}(u,m)}{\mathbf{cn}(u,m)}\right), i\frac{\mathbf{sn}(u,m)}{\mathbf{cn}(u,m)}\right]$

The exponential Riordan array of the derivative subgroup corresponding to (2.3) is $\left[\frac{d}{du}\left(i\frac{\mathrm{sn}(u,m)}{\mathrm{cn}(u,m)}\right), i\frac{\mathrm{sn}(u,m)}{\mathrm{cn}(u,m)}\right] \equiv \left[\frac{d}{du}\mathrm{sn}(iu,k'), \mathrm{sn}(iu,k')\right]$ having coefficient matrix given by

$$A = \begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -i(m-2) & 0 & -i & 0 & 0 \\ 0 & 4(m-2) & 0 & 1 & 0 \\ i\left(m^2-16m+16\right) & 0 & 10i(m-2) & 0 & i \end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & i & 0 & 0 \\ i(m-2) & 0 & i & 0 \\ 0 & 3i(m-2) & 0 & i \\ -3im^2 & 0 & 6i(m-2) & 0 \end{pmatrix}.$$

The production matrices from B in terms of m are as follows:

$$D = \left\{ \begin{pmatrix} 0 & i & 0 & 0 \\ -3i & 0 & i & 0 \\ 0 & -9i & 0 & i \\ -3i & 0 & -18i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 \\ -2i & 0 & i & 0 \\ 0 & -6i & 0 & i \\ 0 & 0 & -12i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & i & 0 \\ 0 & -3i & 0 & i \\ -3i & 0 & -6i & 0 \end{pmatrix} \right\}.$$

2.5.2 The Riordan array from cn(iu, k')

The transformation of the imaginary cosine is given by

$$\mathrm{cn}(iu, k') = \frac{1}{\mathrm{cn}(u,k)} \equiv \mathrm{nc}(u,k)$$

which has the series expansion given by

$$\frac{1}{\mathrm{cn}(u,k)} = 1 + \frac{u^2}{2} + \left(\frac{5}{24} - \frac{k}{6}\right) u^4 + \frac{1}{720}\left(16k^2 - 76k + 61\right) u^6 + \frac{\left(-64k^3 + 1104k^2 - 2424k + 1385\right) u^8}{40320} + O\left(u^9\right) \quad (2.4)$$

The exponential Riordan array corresponding to (2.4) of the the Appell subgroup defined by

$$\left[\frac{1}{\mathrm{cn}(u,k)}, u\right]$$

is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 5-4k & 0 & 6 & 0 & 1 & 0 & 0 \\ 0 & 25-20k & 0 & 10 & 0 & 1 & 0 \\ 16k^2-76k+61 & 0 & 75-60k & 0 & 15 & 0 & 1 \end{pmatrix}.$$

REMARK

- The row sums for case $k = 0$ is given by $\{1, 1, 2, 4, 12, 36, 152, ...\}$ corresponds to **A003701** and having e.g.f $\frac{e^u}{\cos(u)}$.

- The row sums for the case $k = 1$ is given by $\{1, 1, 2, 4, 8, 16, 32, ...\}$ corresponds to **A011782** and has the e.g.f $\cosh(u)e^u = \frac{e^{2u}+1}{2}$.

The production matrix of A in terms of k is given by

$$B=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 2-4k & 0 & 3 & 0 & 1 & 0 \\ 0 & 8-16k & 0 & 4 & 0 & 1 \\ 16\left(k^2-k+1\right) & 0 & 20-40k & 0 & 5 & 0 \end{pmatrix}.$$

The production matrix B for the cases $k=-1,0,1$ is given by

$$\left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 6 & 0 & 3 & 0 & 1 & 0 \\ 0 & 24 & 0 & 4 & 0 & 1 \\ 48 & 0 & 60 & 0 & 5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 8 & 0 & 4 & 0 & 1 \\ 16 & 0 & 20 & 0 & 5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 & 1 & 0 \\ 0 & -8 & 0 & 4 & 0 & 1 \\ 16 & 0 & -20 & 0 & 5 & 0 \end{pmatrix}\right\}.$$

2.5.3 The Riordan array from dn(iu, k')

The imaginary transformation of elliptic dn is

$$\mathrm{dn}(iu,k')=\frac{\mathrm{dn}(u,k)}{\mathrm{cn}(u,k)}\equiv \mathrm{dc}(u,k)$$

and has the series expansion given by

$$1+\left(\frac{1}{2}-\frac{k}{2}\right)u^2+\frac{1}{24}\left(k^2-6k+5\right)u^4+\frac{1}{720}\left(-k^3+47k^2-107k+61\right)u^6+ \frac{\left(k^4-412k^3+2142k^2-3116k+1385\right)u^8}{40320}+O\left(u^9\right). \quad (2.5)$$

The exponential Riordan array corresponding to (2.5) is defined by

$$\left[\frac{\mathrm{dn}(u,m)}{\mathrm{cn}(u,m)},u\right]\equiv[\mathrm{dn}(iu,k),iu]$$

and has the coefficient matrix given by

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2\left(\frac{1}{2}-\frac{m}{2}\right) & 0 & 1 & 0 & 0 \\ 0 & 6\left(\frac{1}{2}-\frac{m}{2}\right) & 0 & 1 & 0 \\ m^2-6m+5 & 0 & 12\left(\frac{1}{2}-\frac{m}{2}\right) & 0 & 1 \end{pmatrix}$$

which is equivalent to

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1-m & 0 & 1 & 0 & 0 \\ 0 & 3-3m & 0 & 1 & 0 \\ m^2-6m+5 & 0 & 6-6m & 0 & 1 \end{pmatrix}.$$

Remark:

- The sequence of numbers $2, 6, 12, ...$ which are located on the $n+2, n$ diagonal of A correspond to **A002378.**

- The row sums of A form the sequence $(1, 1, 2, 4, 12, 36, 152, ...)$ corresponds to **A003701** with e.g.f $\frac{e^u}{\cos(u)}$.

- The row sums of A forming the sequence $(1, 1, 1, 1, 1, 1, 1, 1, ...)$ correspond to **A000012** with e.g.f e^u.

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-m & 0 & 1 & 0 \\ 0 & 2-2m & 0 & 1 \\ 2-2m^2 & 0 & 3-3m & 0 \end{pmatrix}.$$

The production matrices from B in terms of $m = -1, 0, 1$ are as follows:

$$\left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

2.5.4 The Riordan array from cd(iu, k')

The transformation of the imaginary elliptic Jacobi cd is given by $\text{cd}(iu, k') = \frac{1}{\text{dn}(u,k)} \equiv nd(u, k)$ which has the series expansion such that

$$\frac{1}{\text{dn}(u,k)} = 1 + \frac{ku^2}{2} + \left(\frac{5k^2}{24} - \frac{k}{6}\right) u^4 + \frac{1}{720}\left(61k^3 - 76k^2 + 16k\right) u^6 + \frac{\left(1385k^4 - 2424k^3 + 1104k^2 - 64k\right) u^8}{40320} + O\left(u^9\right) \quad (2.6)$$

The exponential Riordan array corresponding to (2.6) is $\left[\frac{1}{\mathrm{dn}(u,k)}, u\right]$ having coefficient matrix given by

$$A=\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ k & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3k & 0 & 1 & 0 & 0 & 0 \\ k(5k-4) & 0 & 6k & 0 & 1 & 0 & 0 \\ 0 & 5k(5k-4) & 0 & 10k & 0 & 1 & 0 \\ k\left(61k^2-76k+16\right) & 0 & 15k(5k-4) & 0 & 15k & 0 & 1 \end{array}\right).$$

REMARK:

- The row sums of A for $k=0$ are $\{1,1,1,1,1,1,1,...\}$ corresponding to **A000012**
- The row sums for $k=1$ given by $\{1,1,2,4,8,16,32\}$ corresponds to **A011782**.

The production matrix of A in terms of k is given by

$$B=\left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ k & 0 & 1 & 0 & 0 & 0 \\ 0 & 2k & 0 & 1 & 0 & 0 \\ 2(k-2)k & 0 & 3k & 0 & 1 & 0 \\ 0 & 8(k-2)k & 0 & 4k & 0 & 1 \\ 16k\left(k^2-k+1\right) & 0 & 20(k-2)k & 0 & 5k & 0 \end{array}\right).$$

The production matrix of B for $k=-1,0,1$ is given by

$$\left\{\left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 6 & 0 & -3 & 0 & 1 & 0 \\ 0 & 24 & 0 & -4 & 0 & 1 \\ -48 & 0 & 60 & 0 & -5 & 0 \end{array}\right),\left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),\left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 & 1 & 0 \\ 0 & -8 & 0 & 4 & 0 & 1 \\ 16 & 0 & -20 & 0 & 5 & 0 \end{array}\right)\right\}.$$

2.6 Jacobi Riordan arrays from the A and Z generating functions

In general an exponential Riordan array $[g(x), f(x)]$ can be expressed in terms of its A and Z generating functions such that

$$[g(x), f(x)] = \left[e^{\int_0^{Rev\left(\int_0^x \frac{dt}{\mathrm{A}(t)}\right)} \frac{\mathrm{Z}(t)}{\mathrm{A}(t)}dt}, Rev\int_0^x \frac{dt}{\mathrm{A}(t)}\right]$$

Example

$$\text{Suppose that } \mathrm{A}(t) = \sqrt{(1-t^2)(1-k^2t^2)}.$$

$$\text{Then } \; Rev\left(\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right)$$

$$= \mathrm{sn}(x)$$

$$\text{So } \; f(x) = \mathrm{sn}(x)$$

$$g(x) = e^{\int_0^{\mathrm{sn}(x)} \frac{\mathrm{Z}(t)}{\mathrm{A}(t)} dt}$$

$$\text{For instance if } \; \mathrm{Z}(t) = \sqrt{1-k^2t^2} \; \text{ then we have}$$

$$\begin{aligned} g(x) &= e^{\int_0^{\mathrm{sn}(x)} \frac{1}{\sqrt{1-t^2}} dt} \\ &= e^{\sin^{-1}(\mathrm{sn}(x))} \end{aligned}$$

$$\text{So } \; [g(x), f(x)] = \left[e^{\sin^{-1}(\mathrm{sn}(x))}, \mathrm{sn}(x)\right]$$

The coefficient array of $\left[e^{\sin^{-1}(\mathrm{sn}(x))}, \mathrm{sn}(x)\right]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1-m & 2-m & 3 & 1 & 0 & 0 & 0 \\ 1-4m & -8m & 2-4m & 4 & 1 & 0 & 0 \\ m^2-6m+1 & m^2-16m-4 & -10(3m+1) & -10m & 5 & 1 & 0 \\ 16m^2+4m+1 & 8\left(4m^2+6m-1\right) & 16m^2-16m-29 & -40(2m+1) & -5(4m+1) & 6 & 1 \end{pmatrix}.$$

Remark The row sums of A at $m = 0$ corresponds to **A009282** having e,g,f $e^{(z+\sin(z))}$.

The production matrix of A is given by

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -m & -m-1 & 1 & 1 & 0 & 0 \\ 0 & -3m & -3(m+1) & 1 & 1 & 0 \\ -3m^2 & -3(m-1)^2 & -6m & -6(m+1) & 1 & 1 \\ 0 & -15m^2 & -15(m-1)^2 & -10m & -10(m+1) & 1 \end{pmatrix}.$$

The production matrix B at $m = -1, 0, 1$ is given by

$$\left\{ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 & 0 \\ -3 & -12 & 6 & 0 & 1 & 1 \\ 0 & -15 & -60 & 10 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & 1 & 1 & 0 \\ 0 & -3 & 0 & -6 & 1 & 1 \\ 0 & 0 & -15 & 0 & -10 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -6 & 1 & 1 & 0 \\ -3 & 0 & -6 & -12 & 1 & 1 \\ 0 & -15 & 0 & -10 & -20 & 1 \end{pmatrix} \right\}$$

Similarly if,

$$\mathrm{Z}(t) = \sqrt{1 - t^2}$$

then we get,

$$g(x) = e^{\frac{1}{k} \sin^{-1}(k \mathrm{sn}(x))}$$

So

$$[g(x), f(x)] = \left[e^{\frac{1}{k} \sin^{-1}(k \mathrm{sn}(x))}, \mathrm{sn}(x) \right]$$

The coefficient matrix of $\left[e^{\frac{1}{k} \sin^{-1}(k \mathrm{sn}(x))}, \mathrm{sn}(x) \right]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2-m & 3 & 1 & 0 & 0 & 0 \\ -3 & -4(m+1) & 2-4m & 4 & 1 & 0 & 0 \\ 4(m-2) & m^2+4m-24 & -20(m+1) & -10m & 5 & 1 & 0 \\ 24m-3 & 6\left(m^2+18m-7\right) & 16m^2+44m-89 & -60(m+1) & -5(4m+1) & 6 & 1 \end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & -m-1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3(m+1) & 1 & 1 & 0 \\ -3 & -3(m-1)^2 & -6 & -6(m+1) & 1 & 1 \\ 0 & -15 & -15(m-1)^2 & -10 & -10(m+1) & 1 \end{pmatrix}.$$

The production matrix B if $m = -1, 0, 1$ is given by

$$\left\{ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 1 & 0 \\ -3 & -12 & -6 & 0 & 1 & 1 \\ 0 & -15 & -60 & -10 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & 1 & 1 & 0 \\ -3 & -3 & -6 & -6 & 1 & 1 \\ 0 & -15 & -15 & -10 & -10 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -6 & 1 & 1 & 0 \\ -3 & 0 & -6 & -12 & 1 & 1 \\ 0 & -15 & 0 & -10 & -20 & 1 \end{pmatrix} \right\}.$$

Example Let us consider the exponential Riordan array $M = [\mathrm{cn}(x), \mathrm{sn}(x)]$

where we suppress the parameter k. We then have

$$\bar{f}(x) = \mathrm{sn}^{-1}(x).$$

We also have $g'(x) = \mathrm{cn}'(x) = -\mathrm{sn}(x)\mathrm{dn}(x)$, so that we obtain

$$\begin{aligned} \mathrm{Z}(x) &= \mathrm{Z}_M(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x)} \\ &= \frac{-\mathrm{sn}\left(\mathrm{sn}^{-1}(x)\right)\mathrm{dn}\left(\mathrm{sn}^{-1}(x)\right)}{\mathrm{cn}\left(\mathrm{sn}^{-1}(x)\right)} \\ &= \frac{-x\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}. \end{aligned}$$

Thus for the exponential Riordan array $[\mathrm{cn}(x,k), \mathrm{sn}(x,k)]$ we have

$$\mathrm{A}(x) = \sqrt{(1-x^2)(1-k^2x^2)}, \quad \mathrm{Z}(x) = \frac{-x\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}.$$

This means in particular that the bivariate generating function of the production matrix of $[\mathrm{cn}(x), \mathrm{sn}(x)]$ is given by

$$e^{xy}\left(\frac{-x\sqrt{1-k^2x^2}}{\sqrt{1-x^2}} + y\sqrt{(1-x^2)(1-k^2x^2)}\right).$$

In this case, we have

$$\frac{\mathrm{Z}(t)}{\mathrm{A}(t)} = \frac{-t\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

or

$$\frac{Z(t)}{A(t)} = \frac{-t}{1-t^2}.$$

Thus

$$\begin{aligned} e^{\int_0^{\mathrm{Rev}\left(\int_0^x \frac{dt}{\mathrm{A}(t)}\right)} \frac{\mathrm{Z}(t)}{\mathrm{A}(t)}\,dt} &= e^{\int_0^{\mathrm{sn}(x,k)} \frac{-t}{1-t^2}\,dt} \\ &= \sqrt{1-\mathrm{sn}(x,k)^2} \\ &= \mathrm{cn}(x,k), \end{aligned}$$

as expected.

To calculate the inverse array $[\mathrm{cn}(x), \mathrm{sn}(x)]$ we have

$$\frac{1}{g(\bar{f}(x))} = \frac{1}{\mathrm{cn}(\mathrm{sn}^{-1}(x))} = \frac{1}{\sqrt{1-x^2}},$$

and so

$$[\mathrm{cn}, \mathrm{sn}(x)]^{-1} = \left[\frac{1}{\sqrt{1-x^2}}, \int_0^x \frac{dt}{\sqrt{(1-k^2t^2)(1-t^2)}}\right].$$

Example We next look at the exponential Riordan array

$$\left[\frac{\mathrm{cn}(x)}{1+\mathrm{sn}(x)}, \mathrm{sn}(x)\right].$$

We once again have

$$\mathrm{A}(x) = \sqrt{(1-x^2)(1-k^2x^2)}.$$

Now $g(x) = \frac{\mathrm{cn}(x)}{1+\mathrm{sn}(x)}$, and we find that

$$g'(x) = \frac{-(1+\mathrm{sn}(x))\mathrm{sn}(x)\mathrm{dn}(x) - \mathrm{cn}(x)^2\mathrm{dn}(x)}{(1+\mathrm{sn}(x))^2}.$$

We then get

$$\begin{aligned} \frac{g'(\mathrm{sn}^{-1}(x)}{g(\mathrm{sn}^{-1})} &= \frac{-(1+x)x\sqrt{1-k^2x^2} - (1-x^2)\sqrt{1-k^2x^2}}{(1+x)^2} \frac{1+x}{\sqrt{1-x^2}} \\ &= -\frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}. \end{aligned}$$

Thus the bivariate generating function for the production matrix of $\left[\frac{\mathrm{cn}(x)}{1+\mathrm{sn}(x)}, \mathrm{sn}(x)\right]$ is given by

$$e^{xy}\left(-\sqrt{\frac{1-k^2x^2}{1-x^2}} + y\sqrt{(1-x^2)(1-k^2x^2)}\right).$$

We note that for $k^2=1$, we get the exponential Riordan array

$$\left[\frac{\mathrm{sech}(x)}{1+\tanh(x)}, \tanh(x)\right].$$

The inverse matrix is calculated as follows.

$$\left[\frac{\mathrm{cn}(x)}{1+\mathrm{sn}(x)}, \mathrm{sn}(x)\right]^{-1} = \left[\frac{1}{\frac{\mathrm{cn}(\mathrm{sn}^{-1}(x)}{1+\mathrm{sn}(\mathrm{sn}^{-1}(x))}}, \mathrm{sn}^{-1}(x)\right]$$
$$= \left[\frac{1+x}{\sqrt{1-x^2}}, \int_0^x \frac{dt}{\sqrt{(1-k^2t^2)(1-t^2)}}\right].$$

Example Our next example is the exponential Riordan array

$$\left[\frac{\mathrm{cn}(x)}{1+\mathrm{sn}(x)}, \frac{\mathrm{sn}(x)}{1+\mathrm{sn}(x)}\right].$$

We do not immediately know what the inverse function of $\frac{\mathrm{sn}(x)}{1+\mathrm{sn}(x)}$ is, so we use the theory of Riordan arrays to continue the analysis.

Thus we note that

$$\left[\frac{\mathrm{cn}(x)}{1+\mathrm{sn}(x)}, \frac{\mathrm{sn}(x)}{1+\mathrm{sn}(x)}\right] = [\mathrm{cn}(x), \mathrm{sn}(x)]\left[\frac{1}{1+x}, \frac{x}{1+x}\right],$$

where the second Riordan array in the product is related to the Laguerre polynomials.

Taking inverses, we obtain

$$\left[\frac{\mathrm{cn}(x)}{1+\mathrm{sn}(x)}, \frac{\mathrm{sn}(x)}{1+\mathrm{sn}(x)}\right]^{-1} = \left[\frac{1}{1+x}, \frac{x}{1+x}\right]^{-1}[\mathrm{cn}(x), \mathrm{sn}(x)]^{-1},$$

or

$$\left[\frac{\mathrm{cn}(x)}{1+\mathrm{sn}(x)}, \frac{\mathrm{sn}(x)}{1+\mathrm{sn}(x)}\right]^{-1} = \left[\frac{1}{1-x}, \frac{x}{1-x}\right]\left[\frac{1}{\sqrt{1-x^2}}, \int_0^x \frac{dt}{(1-t^2)(1-k^2t^2)}\right].$$

This gives us

$$\left[\frac{\mathrm{cn}(x)}{1+\mathrm{sn}(x)}, \frac{\mathrm{sn}(x)}{1+sn(x)}\right]^{-1} = \left[\frac{1}{\sqrt{1-2x}}, \int_0^{\frac{x}{1-x}} \frac{dt}{(1-t^2)(1-k^2t^2)}\right].$$

Thus in particular, we have that

$$\frac{\mathrm{sn}(x)}{1+\mathrm{sn}(x)} = \mathrm{Rev}\int_0^{\frac{x}{1-x}} \frac{dt}{(1-t^2)(1-k^2t^2)},$$

or by the change of variable $y = \frac{t}{1+t}$, we get

$$\frac{\operatorname{sn}(x)}{1+\operatorname{sn}(x)} = \operatorname{Rev} \int_0^x \frac{dy}{\sqrt{(1-2y)(1-2y-(k^2-1)y^2)}}.$$

We can generalize this to the following.

We assume given two exponential Riordan arrays, $M = [g, f]$, and $N = [u, v]$. We assume that M is an elliptic Riordan array, with

$$A_M(t) = R(t, \sqrt{P(t)}).$$

We consider the product

$$M \cdot N = [g, f] \cdot [u, v] = [gu(f), v(f)].$$

Knowing that

$$f(x) = \operatorname{Rev} \int_0^x \frac{dt}{A(t)},$$

we wish to find an "elliptic" characterisation of $v(f)$.

For this, we look at the inverse

$$(M \cdot N)^{-1} = N^{-1} \cdot M^{-1} = \left[\frac{1}{u(\bar{v})}, \bar{v}\right] \cdot \left[\frac{1}{g(\bar{f})}, \bar{f}\right].$$

Now

$$\bar{f}(x) = \int_0^x \frac{dt}{\mathrm{A}(t)},$$

so we obtain

$$(M \cdot N)^{-1} = \left[\frac{1}{u(\bar{v})} \frac{1}{g(\bar{f}(\bar{v}))}, \int_0^{\bar{v}(x)} \frac{dt}{\mathrm{A}(t)}\right].$$

Thus we have that

$$v(f) = \operatorname{Rev} \int_0^{\bar{v}(x)} \frac{dt}{\mathrm{A}(t)}.$$

To put this in an "elliptic" form, we use the change of variable

$$y = v(t) \Longrightarrow t = \bar{v}(y).$$

in the integral.

This gives us

$$\frac{dy}{dt} = v'(t) \Longrightarrow dt = \frac{dy}{v'(t)} = \frac{dy}{v'(\bar{v}(y))} = \bar{v}'(y)dy.$$

When $t = \bar{v}(x)$, we have $y = v(t) = v(\bar{v}(x)) = x$, and so we have

$$\int_0^{\bar{v}(x)} \frac{dt}{\mathrm{A}(t)} = \int_0^x \frac{\bar{v}'(y)dy}{\mathrm{A}(\bar{v}(y))}.$$

Thus we have

$$v(f) = \mathrm{Rev} \int_0^x \frac{\bar{v}'(y)dy}{\mathrm{A}(\bar{v}(y))} = \mathrm{Rev} \int_0^x \frac{dy}{v'(\bar{v}(y))\mathrm{A}(\bar{v}(y))}.$$

Example In this example, we seek to write the elliptic function

$$\frac{\mathrm{sn}(x)(1+\mathrm{sn}(x))}{1-\mathrm{sn}(x)}$$

as the reversion of an integral whose limits are 0 to x. For this, we consider the elliptic Riordan array $[\mathrm{cn}(x), \mathrm{sn}(x)]$ and the transformation given by the exponential Riordan array $\left[\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right]$.

Thus we have the product

$$[\mathrm{cn}(x), \mathrm{sn}(x)] \cdot \left[\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right] = \left[\frac{\mathrm{cn}(x)}{1-\mathrm{sn}(x)}, \frac{\mathrm{sn}(x)(1+\mathrm{sn}(x))}{1-\mathrm{sn}(x)}\right].$$

Here,

$$\left[\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right]^{-1} = \left[1 - \frac{\sqrt{1+6x+x^2}-x-1}{2}, \frac{\sqrt{1+6x+x^2}-x-1}{2}\right].$$

In particular,

$$\bar{v}(x) = \frac{\sqrt{1+6x+x^2}-x-1}{2},$$

and

$$\bar{v}'(x) = \frac{3+x-\sqrt{1+6x+x^2}}{2\sqrt{1+6x+x^2}}.$$

Also,

$$\mathrm{sn}(x) = \mathrm{Rev} \int_0^x \frac{dt}{(1-k^2t^2)(1-t^2)}.$$

Thus we have

$$\frac{\operatorname{sn}(x)(1+\operatorname{sn}(x))}{1-\operatorname{sn}(x)} = Rev \int_0^x \frac{\frac{3+y-\sqrt{1+6y+y^2}}{2\sqrt{1+6y+y^2}}dy}{\sqrt{\left(1-k^2\left(\frac{\sqrt{1+6y+y^2}-y-1}{2}\right)^2\right)\left(1-\left(\frac{\sqrt{1+6y+y^2}-y-1}{2}\right)^2\right)}}.$$

Chapter 3

Weierstrass Elliptic Functions and Riordan Arrays

3.1 Introduction

The previous chapter treated the Jacobi elliptic functions and the construction of Riordan arrays from these functions to form the Jacobi Riordan arrays. The Jacobi elliptic functions have a simple analytical relation to the Weierstrass $\wp$ function. The link between the Weierstrass $\wp$ function and the Jacobi elliptic function is given explicitly by

$$\wp(z) = \frac{1}{\mathrm{sn}(z,m)^2} - \frac{m+1}{3}$$

where m is the elliptic modulus.

The main purpose of this chapter is to establish the relationship between the Weierstrass elliptic $\wp$ functions and Riordan arrays forming new lower triangular matrices known as Weierstrass Riordan arrays. The relationship between the Weierstrass $\wp$ functions and Riordan arrays can be examined based on the Taylor series expansion of these functions in both the cases of the elliptic modulus m and their invariants $\{g_2, g_3\}$.

The origins of the Weierstrass $\wp$ functions as earlier discussed in section (1.8) can be traced back to the 19th century pioneering work of two famous mathematicians N.H Abel(1827) and K. Weierstrass(1855,1862). The first known Weierstrass function was referred to as the "Weierstrass P". Further developments of the subject from G. Eisentein (1847) and K. Weierstrass (1855,1862,1895) led to the formulation of the sigma and zeta Weierstrass functions [118].

The Weierstrass elliptic function $\wp(z)$ [45] can be defined by an infinite sum on the complex plane over a period lattice such that

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\in\mathbb{Z}} \left\{ \frac{1}{(z-2m\omega_1-2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2} \right\},$$

where ω_1 and ω_2 are called the *half periods* and $m, n \neq 0$. The periods of $\wp$ are $2K$ and $2iK'$ where K' is defined as K but using $k' = \sqrt{1-k^2}$ instead of k. Let $\omega = K$ and $\omega' = iK'$, be the half periods of $\wp$ and set

$$e_1 = \wp(\omega),$$

$$e_2 = \wp(\omega'),$$

$$e_3 = \wp(\omega''),$$

where

$$\omega + \omega' = \omega''.$$

The Weierstrass elliptic function $\wp(z)$ satisfies the differential equation

$$\wp'(z,k)^2 = 4(\wp(z,k)-e_1)(\wp(z,k)-e_2)(\wp(z,k)-e_3).$$

The Weierstrass $\wp$ elliptic function denoted $\wp(z; g_2, g_3)$ depends on the argument z and the two parameters $\{g_2, g_3\}$. A meromorphic function is considered to be a Weierstrass elliptic function $\wp$ if it is doubly periodic with two periods usually denoted by 2ω and $2\omega'$ such that $\wp(z+2\omega) = \wp(z+2\omega') = \wp(z)$ and it satisfies the differential equation

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3$$

and

$$z = \int_{t=w}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} \text{ s.t } w = \wp(z)$$

where $g_2 = 60\sum' \frac{1}{(2m\omega_1+2n\omega_2)^4}$ & $g_3 = 140\sum' \frac{1}{(2m\omega_1+2n\omega_2)^6}$

with the prime ($\prime$) indicating summation over $\mathbb{Z}^2$ excluding $(m,n) = (0,0)$ [45].

Furthermore, the Weierstrass elliptic function $\wp$ and its derivative $\wp'$ parameterizes the elliptic curve over complex numbers.

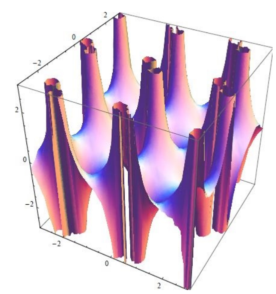


Figure 3.1: Plot depicting a Weierstrass $\wp$ over a complex plane with $g_3 = 1+i$

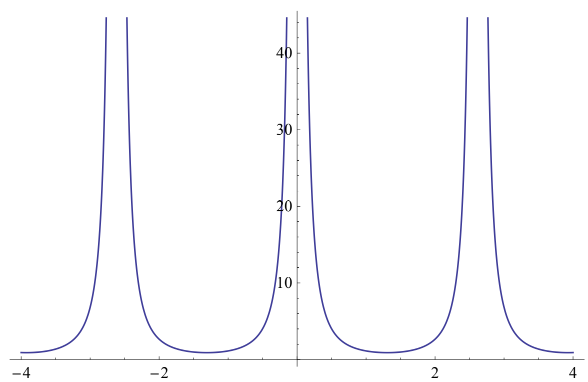


Figure 3.2: The plot of Weierstrass $\wp$ elliptic function having $\{g_2, g_3\} = \{1, 2\}$

The Weierstrass elliptic function $\wp$ is the most basic Weierstrass elliptic function from which the other types of Weierstrass pseudo- elliptic functions can be constructed. The other two most important types of Weierstrass pseudo-elliptic functions are the Weierstrass zeta function (ζ) and Weierstrass sigma function (σ). The Weierstrass zeta ζ function is an odd function such that

$$\zeta(-z) = \zeta(z) \qquad \& \qquad \zeta'(z) = -\wp(z).$$

The Weierstrass zeta ζ function is quasi-periodic such that [125]

$$\zeta(z+2\omega_k) = \zeta(z) + 2\eta_k, k = 1, 2, 3$$

.

The Weierstrass sigma function is also an odd function such that

$$\sigma(-z) = -\sigma(z) \qquad \& \qquad \frac{\sigma'(z)}{\sigma(z)} = \zeta(z).$$

The Weierstrass σ is quasi-periodic such that

$$\sigma(z+2\omega_k) = -e^{(2\eta_k(z+\omega_k))\sigma(z)}.$$

The Weierstrass σ function is connected to the Weierstrass $\wp$ function by the formula

$$\frac{\sigma(x+y)\sigma(x-y)}{\sigma^2(x)\sigma^2(y)} = \wp(y) - \wp(x).$$

The Taylor series expansion around $z = 0$ of the Weierstrass elliptic functions in terms of g_2, g_3 are listed below.

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \frac{g_2^2}{1200}z^6 + \frac{3g_2g_3}{6160}z^8 + \left(\frac{g_2^3}{156000} + \frac{g_3^2}{10192}\right)z^{10} + \frac{g_2^2g_3}{184800}z^{12} + O\left(z^{13}\right).$$

$$\wp'(z) = -\frac{2}{z^3} + \frac{g_2}{10}z + \frac{g_3}{7}z^3 + \frac{g_2^2}{200}z^5 + \frac{3g_2g_3}{770}z^7 + \left(\frac{g_2^3}{15600} + \frac{5g_3^2}{5096}\right)z^9 + \frac{g_2^2g_3}{15400}z^{11} + O\left(z^{13}\right).$$

$$\zeta(z) = \frac{1}{z} - \frac{g_2}{60}z^3 - \frac{g_3}{140}z^5 - \frac{g_2^2}{8400}z^7 - \frac{g_2g_3}{18480}z^9 + \left(-\frac{g_2^3}{1716000} - \frac{g_3^2}{112112}\right)z^{11} + O\left(z^{13}\right).$$

$$\sigma(z) = z - \frac{g_2}{240}z^5 - \frac{g_3}{840}z^7 - \frac{g_2^2}{161280}z^9 - \frac{g_2 g_3}{2217600}z^{11} + O\left(z^{13}\right).$$

The invariants g_2 and g_3 can be expressed in terms of m such that we have:

$$\begin{aligned} g_2 &= \frac{4}{3}\left(m^2 - m + 1\right) \\ g_3 &= \frac{4}{27}\left(2m^3 - 3m^2 - 3m + 2\right). \end{aligned}$$

The corresponding Taylor series expansions of the Weierstrass elliptic functions around $z = 0$ in terms of the modulus m are listed below.

$$\wp(z) = \frac{1}{z^2} + \frac{1}{15}\left(m^2 - m + 1\right)z^2 + \frac{1}{189}\left(2m^3 - 3m^2 - 3 + 2\right)z^4 + \\ \frac{1}{675}\left(m^4 - 2m^3 + 3m^2 - 2m + 1\right)z^6 + O\left(z^8\right).$$

$$\zeta(z) = -\frac{\left(m^2 - m + 1\right)^2 z^7}{4725} - \frac{1}{45}\left(m^2 - m + 1\right)z^3 - \\ \frac{1}{945}\left(2m^3 - 3m^2 - 3m + 2\right)z^5 + \frac{1}{z}.$$

$$\sigma(z) = 1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \frac{z^4}{24} + \frac{\left(8m^2 - 8m - 7\right)z^5}{1800} + \frac{\left(-16m^2 + 16m - 11\right)z^6}{3600} + \\ O\left(z^7\right).$$

3.2 Examples of Weierstrass Riordan arrays

We provide several coefficient matrices which define Riordan arrays based on the generating functions of Weierstrass functions together with their corresponding production matrices. The Riordan arrays are computed seperately in terms of either the invariants $\{g_2, g_3\}$ and the elliptic modulus m.
The first set of examples below give the Riordan arrays expressed in terms of the *invariants.*

3.2.1 Weierstrass Riordan arrays in terms of the Invariants $\{g_2, g_3\}$

The examples below are Weierstrass Riordan arrays in terms of the invariants.

3.2.1.1 The Riordan array $[1, \sigma(z; \mathbf{g}_2, \mathbf{g}_3)]$

The coefficient array of $[1, \sigma(z; g_2, g_3)]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{g_2}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3g_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -6g_3 & 0 & -\frac{21g_2}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -48g_3 & 0 & -28g_2 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The production matrix of A is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{g_2}{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{5g_2}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & -6g_3 & 0 & -\frac{15g_2}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & -42g_3 & 0 & -\frac{35g_2}{2} & 0 & 0 & 0 \end{pmatrix}.$$

3.2.1.2 The Riordan array $\left[\frac{d}{dz} z^3 \wp(z; \mathbf{g}_2, \mathbf{g}_3), z^3 \wp(z; \mathbf{g}_2, \mathbf{g}_3)\right]$

The coefficient array of $\left[\frac{d}{dz} z^3 \wp(z; g_2, g_3), z^3 \wp(z; g_2, g_3)\right]$ where $\frac{d}{dz} z^3 \wp(z; g_2, g_3) = 3z^2 \wp(z; g_2, g_3) + z^3 \wp'(z; g_2, g_3)$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6g_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 36g_2 & 0 & 0 & 0 & 1 & 0 \\ 180g_3 & 0 & 126g_2 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The production matrix of A is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 6g_2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 30g_2 & 0 & 0 & 0 & 1 \\ 180g_3 & 0 & 90g_2 & 0 & 0 & 0 \end{pmatrix}.$$

3.2.1.3 The Riordan array $\left[\frac{d}{dz}z^2\zeta(z;\mathbf{g}_2,\mathbf{g}_3), z^2\zeta(z;\mathbf{g}_2,\mathbf{g}_3)\right]$

The coefficient array of $\left[\frac{d}{dz}z^2\zeta(z;\mathrm{g}_2,\mathrm{g}_3), z^2\zeta(z;\mathrm{g}_2,\mathrm{g}_3)\right]$ where $\frac{d}{dz}z^2\zeta(z;\mathrm{g}_2,\mathrm{g}_3) = 2z\zeta(z;\mathrm{g}_2,\mathrm{g}_3) - z^2\wp(z;\mathrm{g}_2,\mathrm{g}_3)$ results to

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -12\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -36\mathrm{g}_3 & 0 & -42\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -288\mathrm{g}_3 & 0 & -112\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 \\ -\frac{216{\mathrm{g}_2}^2}{5} & 0 & -1296\mathrm{g}_3 & 0 & -252\mathrm{g}_2 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The production matrix of A is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -10\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ -36\mathrm{g}_3 & 0 & -30\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -252\mathrm{g}_3 & 0 & -70\mathrm{g}_2 & 0 & 0 & 0 & 1 \\ -\frac{1336{\mathrm{g}_2}^2}{5} & 0 & -1008\mathrm{g}_3 & 0 & -140\mathrm{g}_2 & 0 & 0 & 0 \end{pmatrix}.$$

3.2.1.4 The Riordan array $\left[1, z^2\zeta(z;\mathbf{g}_2,\mathbf{g}_3)\right]$

The coefficient array of $\left[1, z^2\zeta(z;\mathrm{g}_2,\mathrm{g}_3)\right]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -12\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -36\mathrm{g}_3 & 0 & -42\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -288\mathrm{g}_3 & 0 & -112\mathrm{g}_2 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The production matrix of A is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -10\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -36\mathrm{g}_3 & 0 & -30\mathrm{g}_2 & 0 & 0 & 0 & 1 \\ 0 & 0 & -252\mathrm{g}_3 & 0 & -70\mathrm{g}_2 & 0 & 0 & 0 \end{pmatrix}.$$

Remark: The diagonal sequence of elements $0, 2, 10, 30, 70, ...$ in B correspond to **A034827.**

3.2.1.5 The Riordan array $\left[\frac{d}{dz}\sigma(z;\mathbf{g}_2,\mathbf{g}_3),\sigma(z;\mathbf{g}_2,\mathbf{g}_3)\right]$

The coefficient array of $\left[\frac{d}{dz}\sigma(z;\mathrm{g}_2,\mathrm{g}_3),\sigma(z;\mathrm{g}_2,\mathrm{g}_3)\right]$ where $\sigma(z;\mathrm{g}_2,\mathrm{g}_3)\zeta(z;\mathrm{g}_2,\mathrm{g}_3)$ is given by

$$A=\left(\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
-\frac{\mathrm{g}_2}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & -3\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
-6\mathrm{g}_3 & 0 & -\frac{21\mathrm{g}_2}{2} & 0 & 0 & 0 & 1 & 0 & 0\\
0 & -48\mathrm{g}_3 & 0 & -28\mathrm{g}_2 & 0 & 0 & 0 & 1 & 0\\
-\frac{9\mathrm{g}_2{}^2}{4} & 0 & -216\mathrm{g}_3 & 0 & -63\mathrm{g}_2 & 0 & 0 & 0 & 1
\end{array}\right)$$

The production matrix of A is given by

$$B=\left(\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
-\frac{\mathrm{g}_2}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & -\frac{5\mathrm{g}_2}{2} & 0 & 0 & 0 & 1 & 0 & 0\\
-6\mathrm{g}_3 & 0 & -\frac{15\mathrm{g}_2}{2} & 0 & 0 & 0 & 1 & 0\\
0 & -42\mathrm{g}_3 & 0 & -\frac{35\mathrm{g}_2}{2} & 0 & 0 & 0 & 1\\
-\frac{65\mathrm{g}_2{}^2}{4} & 0 & -168\mathrm{g}_3 & 0 & -35\mathrm{g}_2 & 0 & 0 & 0
\end{array}\right).$$

3.2.2 Weierstrass Riordan arrays in terms of the Modulus m

In this section the power series expansion of the the 3 main types of Weierstrass functions in terms of m listed in section (3.1) will be used to define Riordan arrays according to their subgroups.

3.2.2.1 Appell Subgroup of Weierstrass Riordan arrays

3.2.2.2 $\left[z^2\wp(z,m),z\right]$

The coefficient array in terms of m of $\left[z^2\wp(z,m),z\right]$ is given by

$$A=\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0\\
\frac{8}{5}\left(m^2-m+1\right) & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 8\left(m^2-m+1\right) & 0 & 0 & 0 & 1 & 0\\
\frac{80}{21}\left(2m^3-3m^2-3m+2\right) & 0 & 24\left(m^2-m+1\right) & 0 & 0 & 0 & 1
\end{array}\right).$$

Remark: The columns of the matrix A are palindromic.

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{8}{5}((m-1)m+1) & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{32}{5}((m-1)m+1) & 0 & 0 & 0 & 1 \\ \frac{80}{21}(m-2)(m+1)(2m-1) & 0 & 16((m-1)m+1) & 0 & 0 & 0 \end{pmatrix}.$$

The production of B in terms of $m = \{-1, 0, 1\}$ is given by

$$C = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{24}{5} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{96}{5} & 0 & 0 & 0 & 1 \\ 0 & 0 & 48 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{8}{5} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{32}{5} & 0 & 0 & 0 & 1 \\ \frac{160}{21} & 0 & 16 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{8}{5} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{32}{5} & 0 & 0 & 0 & 1 \\ -\frac{160}{21} & 0 & 16 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

3.2.2.3 $[z\zeta(z, m), z]$

The coefficient array in terms of m of $[z\zeta(z, m), z]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{8}{15}\left(m^2 - m + 1\right) & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{8}{3}\left(m^2 - m + 1\right) & 0 & 0 & 0 & 1 & 0 \\ \frac{16}{21}\left(-2m^3 + 3m^2 + 3m - 2\right) & 0 & -8\left(m^2 - m + 1\right) & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark: The columns of the matrix A are palindromic.

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{8}{15}((m-1)m+1) & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{32}{15}((m-1)m+1) & 0 & 0 & 0 & 1 \\ -\frac{16}{21}(m-2)(m+1)(2m-1) & 0 & -\frac{16}{3}((m-1)m+1) & 0 & 0 & 0 \end{pmatrix}.$$

The production matrix of B in terms of $m = -1, 0, 1$ is given by

$$C = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{8}{5} & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{32}{5} & 0 & 0 & 0 & 1 \\ 0 & 0 & -16 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{8}{15} & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{32}{15} & 0 & 0 & 0 & 1 \\ -\frac{32}{21} & 0 & -\frac{16}{3} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{8}{15} & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{32}{15} & 0 & 0 & 0 & 1 \\ \frac{32}{21} & 0 & -\frac{16}{3} & 0 & 0 & 0 \end{pmatrix} \right\}.$$

3.2.2.4 Lagrange Subgroup of Weierstrass Riordan arrays

3.2.2.5 $[1, z\sigma(z,m)]$

The coefficient array in terms of m for $[1, z\sigma(z,m)]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 1 & 0 & 0 & 0 \\ 0 & -4 & 24 & -12 & 1 & 0 & 0 \\ 0 & 5 & -80 & 90 & -20 & 1 & 0 \\ 0 & \frac{2}{5}\left(8m^2-8m-7\right) & 240 & -540 & 240 & -30 & 1 \end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & -1 & -4 & 1 & 0 & 0 \\ 0 & -4 & -3 & -6 & 1 & 0 \\ 0 & -27 & -16 & -6 & -8 & 1 \\ 0 & \frac{16}{5}((m-1)m-79) & -135 & -40 & -10 & -10 \end{pmatrix}.$$

The production matrix of B in terms of $m = 0, 1$ is given by

$$C = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & -1 & -4 & 1 & 0 & 0 \\ 0 & -4 & -3 & -6 & 1 & 0 \\ 0 & -27 & -16 & -6 & -8 & 1 \\ 0 & -\frac{1264}{5} & -135 & -40 & -10 & -10 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & -1 & -4 & 1 & 0 & 0 \\ 0 & -4 & -3 & -6 & 1 & 0 \\ 0 & -27 & -16 & -6 & -8 & 1 \\ 0 & -\frac{1264}{5} & -135 & -40 & -10 & -10 \end{pmatrix} \right\}.$$

3.2.2.6 $[1, z\zeta(z,m)]$

The coefficient array in terms of m for $[1, z\zeta(z,m)]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{8}{3}\left(m^2-m+1\right) & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -16\left(m^2-m+1\right) & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{8}{3}((m-1)m+1) & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{40}{3}((m-1)m+1) & 0 & 0 & 0 \end{pmatrix}.$$

The production matrix of B in terms of $m = -1, 0, 1$ is given by

$$C = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -8 & 0 & 0 & 0 & 1 \\ 0 & 0 & -40 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{8}{3} & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{40}{3} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{8}{3} & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{40}{3} & 0 & 0 & 0 \end{pmatrix} \right\}.$$

3.2.2.7 Derivative subgroup of Weierstrass Riordan arrays

3.2.2.8 $\left[\frac{d}{dz} z^3 \wp(z,m), z^3 \wp(z,m)\right]$

The coefficient array in terms of m of $\left[\frac{d}{dz} z^3 \wp(z,m), z^3 \wp(z,m)\right]$ where

$$\frac{d}{dz} z^3 \wp(z,m) = 3z^2 \left(\frac{1}{\mathrm{sn}(z|m)^2} + \frac{1}{3}(-m-1) \right) - \frac{2z^3 \mathrm{cn}(z|m)\mathrm{dn}(z|m)}{\mathrm{sn}(z|m)^3}$$

is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 8\left(m^2 - m + 1\right) & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 48\left(m^2 - m + 1\right) & 0 & 0 & 0 & 1 & 0 \\ \frac{80}{3}\left(2m^3 - 3m^2 - 3m + 2\right) & 0 & 168\left(m^2 - m + 1\right) & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It can be noted that the Riordan array $\left[\frac{d}{dz} z^3 \wp(z,m), z^3 \wp(z,m)\right]$ produces the following generating functions for the case $m = 1$ and $m = 0$ below. If $m = 1$ then $\frac{d}{dz} z^3 \wp(z,m)$ becomes

$$3z^2 \left(\coth^2(z) - \frac{2}{3} \right) - 2z^3 \coth(z) \mathrm{csch}^2(z).$$

If $m = 1$ then $z^3 \wp(z,m)$ becomes

$$z^3 \left(\coth^2(z) - \frac{2}{3} \right).$$

If $m = 0$ then $\frac{d}{dz} z^3 \wp(z,m)$ becomes

$$3z^2 \left(\csc^2(z) - \frac{1}{3} \right) - 2z^3 \cot(z) \csc^2(z).$$

If $m=0$ then $z^3\wp(z,m)$ becomes

$$z^3\left(\csc^2(z)-\frac{1}{3}\right).$$

Remark: The columns of the matrix A are palindromic.

The production matrix in terms of m is given by

$$B=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 8((m-1)m+1) & 0 & 0 & 0 & 1 & 0\\ 0 & 40((m-1)m+1) & 0 & 0 & 0 & 1\\ \frac{80}{3}(m-2)(m+1)(2m-1) & 0 & 120((m-1)m+1) & 0 & 0 & 0 \end{pmatrix}.$$

The production matrix of B in terms of $m=-1,0,1$

$$C=\left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 24 & 0 & 0 & 0 & 1 & 0\\ 0 & 120 & 0 & 0 & 0 & 1\\ 0 & 0 & 360 & 0 & 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 8 & 0 & 0 & 0 & 1 & 0\\ 0 & 40 & 0 & 0 & 0 & 1\\ \frac{160}{3} & 0 & 120 & 0 & 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 8 & 0 & 0 & 0 & 1 & 0\\ 0 & 40 & 0 & 0 & 0 & 1\\ -\frac{160}{3} & 0 & 120 & 0 & 0 & 0 \end{pmatrix}\right\}.$$

3.2.2.9 $\left[\frac{d}{dz}z\sigma(z,m), z\sigma(z,m)\right]$

The coefficient array in terms of m for $\left[\frac{d}{dz}z\sigma(z,m), z\sigma(z,m)\right]$ is given by

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ -2 & 1 & 0 & 0 & 0 & 0 & 0\\ 3 & -6 & 1 & 0 & 0 & 0 & 0\\ -4 & 24 & -12 & 1 & 0 & 0 & 0\\ 5 & -80 & 90 & -20 & 1 & 0 & 0\\ \frac{2}{5}\left(8m^2-8m-7\right) & 240 & -540 & 240 & -30 & 1 & 0\\ -\frac{7}{5}\left(16m^2-16m+11\right) & \frac{112}{5}\left(m^2-m-29\right) & 2835 & -2240 & 525 & -42 & 1 \end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B=\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0\\ -1 & -4 & 1 & 0 & 0 & 0\\ -4 & -3 & -6 & 1 & 0 & 0\\ -27 & -16 & -6 & -8 & 1 & 0\\ \frac{16}{5}((m-1)m-79) & -135 & -40 & -10 & -10 & 1\\ 80(m-1)m-3045 & \frac{96}{5}((m-1)m-79) & -405 & -80 & -15 & -12 \end{pmatrix}.$$

The production matrix of B in terms of $m = 0, 1$ is given by

$$C = \left\{ \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -4 & 1 & 0 & 0 & 0 \\ -4 & -3 & -6 & 1 & 0 & 0 \\ -27 & -16 & -6 & -8 & 1 & 0 \\ -\frac{1264}{5} & -135 & -40 & -10 & -10 & 1 \\ -3045 & -\frac{7584}{5} & -405 & -80 & -15 & -12 \end{pmatrix}, \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -4 & 1 & 0 & 0 & 0 \\ -4 & -3 & -6 & 1 & 0 & 0 \\ -27 & -16 & -6 & -8 & 1 & 0 \\ -\frac{1264}{5} & -135 & -40 & -10 & -10 & 1 \\ -3045 & -\frac{7584}{5} & -405 & -80 & -15 & -12 \end{pmatrix} \right\}.$$

3.2.2.10 $\left[\frac{d}{dz} z\zeta(z,m), z\zeta(z,m)\right]$

The coefficient array in terms of m for $\left[\frac{d}{dz} z\zeta(z,m), z\zeta(z,m)\right]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{8}{3}\left(m^2 - m + 1\right) & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -16\left(m^2 - m + 1\right) & 0 & 0 & 0 & 1 & 0 \\ \frac{16}{3}\left(-2m^3 + 3m^2 + 3m - 2\right) & 0 & -56\left(m^2 - m + 1\right) & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark: The columns of the matrix A are palindromic.

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{8}{3}((m-1)m+1) & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{40}{3}((m-1)m+1) & 0 & 0 & 0 & 1 \\ -\frac{16}{3}(m-2)(m+1)(2m-1) & 0 & -40((m-1)m+1) & 0 & 0 & 0 \end{pmatrix}.$$

The production matrices from B for $m = \{-1, 0, 1\}$ is given by

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -8 & 0 & 0 & 0 & 1 & 0 \\ 0 & -40 & 0 & 0 & 0 & 1 \\ 0 & 0 & -120 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{8}{3} & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{40}{3} & 0 & 0 & 0 & 1 \\ -\frac{32}{3} & 0 & -40 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{8}{3} & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{40}{3} & 0 & 0 & 0 & 1 \\ \frac{32}{3} & 0 & -40 & 0 & 0 & 0 \end{pmatrix}.$$

3.2.2.11 Bell Subgroup of Weierstrass Riordan arrays

3.2.2.12 $[\sigma(z,m), z\sigma(z,m)]$

The coefficient array in terms of m of $[\sigma(z,m), z\sigma(z,m)]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 0 & 0 \\ -1 & 12 & -9 & 1 & 0 & 0 & 0 \\ 1 & -32 & 54 & -16 & 1 & 0 & 0 \\ \frac{1}{15}\left(8m^2-8m-7\right) & 80 & -270 & 160 & -25 & 1 & 0 \\ \frac{1}{5}\left(-16m^2+16m-11\right) & \frac{32}{5}\left(m^2-m-29\right) & 1215 & -1280 & 375 & -36 & 1 \end{pmatrix}.$$

The production matrix of A in terms of m is

$$B = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & 1 & 0 & 0 \\ 0 & -4 & -3 & -7 & 1 & 0 \\ \frac{8}{15}((m-1)m+1) & -27 & -16 & -6 & -9 & 1 \\ \frac{32}{3}((m-1)m+1) & \frac{8}{15}(11(m-1)m-469) & -135 & -40 & -10 & -11 \end{pmatrix}.$$

The production matrix of B inn terms of $m = 0, 1$ is

$$C = \left\{ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & 1 & 0 & 0 \\ 0 & -4 & -3 & -7 & 1 & 0 \\ \frac{8}{15} & -27 & -16 & -6 & -9 & 1 \\ \frac{32}{3} & -\frac{3752}{15} & -135 & -40 & -10 & -11 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & 1 & 0 & 0 \\ 0 & -4 & -3 & -7 & 1 & 0 \\ \frac{8}{15} & -27 & -16 & -6 & -9 & 1 \\ \frac{32}{3} & -\frac{3752}{15} & -135 & -40 & -10 & -11 \end{pmatrix} \right\}.$$

3.2.2.13 $\left[z^2\wp(z,m), z^3\wp(z,m)\right]$

The coefficient array in terms of m of $\left[z^2\wp(z,m), z^3\wp(z,m)\right]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{8}{5}\left(m^2-m+1\right) & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 16\left(m^2-m+1\right) & 0 & 0 & 0 & 1 & 0 \\ \frac{80}{21}\left(2m^3-3m^2-3m+2\right) & 0 & 72\left(m^2-m+1\right) & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark: The columns of the matrix A are palindromic.

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{8}{5}((m-1)m+1) & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{72}{5}((m-1)m+1) & 0 & 0 & 0 & 1 \\ \frac{80}{21}(m-2)(m+1)(2m-1) & 0 & 56((m-1)m+1) & 0 & 0 & 0 \end{pmatrix}.$$

The production matrix of B in terms of $m = -1, 0, 1$ is given by

$$C = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{24}{5} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{216}{5} & 0 & 0 & 0 & 1 \\ 0 & 0 & 168 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{8}{5} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{72}{5} & 0 & 0 & 0 & 1 \\ \frac{160}{21} & 0 & 56 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{8}{5} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{72}{5} & 0 & 0 & 0 & 1 \\ -\frac{160}{21} & 0 & 56 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Chapter 4

Dixonian Elliptic Functions and Riordan arrays

4.1 Introduction

The Dixonian elliptic functions (**A104134**, **A104133**) were first introduced in the seminal work of the English mathematician Alfred Cardew Dixon (1890)[32]. The first significant contribution of the Dixonian elliptic functions in mathematics, was that they parameterized the Fermat cubic curve

$$x^3 + y^3 - 3axy = 1,$$

for the case $a = 0$, for which the functions display a special hexagonal symmetry. In particular, this property corresponds to the case for which $g_2 = 0$ in the Weierstrass theory or the case $g_3 = 0$ for the lemniscate.
The two main types of Dixonian elliptic functions are the *Dixonian sine* sm and the *Dixonian cosine* cm. The Trigonometric Dixonian functions are defined in terms of the $\wp$ and $\wp'$ functions such that

$$\mathrm{cm}(z) = \frac{3\wp'\left(z; 0, \frac{1}{27}\right) + 1}{3\wp'\left(z; 0, \frac{1}{27}\right) - 1}$$

and

$$\mathrm{sm}(z) = \frac{6\wp\left(z; 0, \frac{1}{27}\right)}{1 - 3\wp'\left(z; 0, \frac{1}{27}\right)}.$$

These functions satisfy the non-linear differential system

$$\mathrm{sm}'(z) = \mathrm{cm}(z)^2 \;\&\; \mathrm{cm}'(z) = -\mathrm{sm}(z)^2$$

such that

$$\mathrm{sm}(0) = 0 \qquad \mathrm{cm}(0) = 1$$

and

$$\mathrm{sm}^3(z) + \mathrm{cm}^3(z) = 1.$$

The Dixonian elliptic functions sm and cm have periods $3K$ and $3\omega K$ where

$$\omega = \frac{-1 + i\sqrt{3}}{2}$$

is a cube root of unity, such that for $j = 0, 1, 2, ...$

$$\mathrm{sm}(z + 3\omega^j K) = \mathrm{sm}\ z \qquad \mathrm{cm}(z + 3\omega^j K) = \mathrm{cm}\ z.$$

The Dixonian functions sm and cm as expressed in [26, 65] can explicitly be defined as

$$\mathrm{sm}(z) = Rev \int_0^z \frac{dt}{(1-t^3)^{2/3}} \text{ and } \mathrm{cm}(z) = \left(\int_z^1 \frac{dt}{(1-t^3)^{2/3}} \right)^{-1}. \tag{4.1}$$

Furthermore, Conrad & Flajolet [26] also established the combinatorial and probabilistic significance of sm and cm. This connection from these two important areas of mathematics is based on the unusual continued fraction expansions derived from their corresponding Laplace transforms.

4.2 Dixonian Riordan arrays

In this section we establish the relationship between Dixon elliptic functions and Riordan arrays by constructing proper exponential Riordan arrays based on the Dixon elliptic functions. The Taylor series expansion around $z = 0$ for these functions is given by

$$\mathrm{cm}(z) = 1 - \frac{z^3}{3} + \frac{z^6}{18} - \frac{23z^9}{2268} + \frac{25z^{12}}{13608} + O\left(z^{13}\right) \equiv 1 - 2\frac{z^3}{3!} + 40\frac{z^6}{6!} - 3680\frac{z^9}{9!} + O\left(z^{13}\right)$$

and

$$\mathrm{sm}(z) = z - \frac{z^4}{6} + \frac{2z^7}{63} - \frac{13z^{10}}{2268} + O\left(z^{13}\right) \equiv z - 4\frac{z^4}{4!} + 160\frac{z^7}{7!} - 20800\frac{z^{10}}{10!} + O\left(z^{13}\right).$$

$\mathrm{sm}(z)$ defined in (4.1) is suited exponential Riordan arrays with $\mathrm{A}(z) = \left(1 - z^3\right)^{2/3}$ of the form

$$\left[g(z), Rev \int_0^z \frac{dt}{A(t)}\right] \text{ where } g(z) = \int_0^{\mathrm{sm}(z)} \frac{Z(t)}{A(t)} dt$$

for suitable $Z(t)$.

4.2.1 The Riordan array $\left[\frac{d}{dz}\mathbf{sm}(z), \mathbf{sm}(z)\right] \equiv \left[\mathbf{cm}(z)^2, \mathbf{sm}(z)\right]$

The coefficient array of $\left[\frac{d}{dz}\mathrm{sm}(z), \mathrm{sm}(z)\right]$ is given by

$$\mathrm{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -60 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 160 & 0 & 0 & -140 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1840 & 0 & 0 & -280 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 10800 & 0 & 0 & -504 & 0 & 0 & 1 & 0 & 0 \\ -20800 & 0 & 0 & 44400 & 0 & 0 & -840 & 0 & 0 & 1 & 0 \\ 0 & -440000 & 0 & 0 & 145200 & 0 & 0 & -1320 & 0 & 0 & 1 \end{pmatrix}.$$

The production matrix of M is given by

$$\mathrm{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -16 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -40 & 0 & 0 & 1 & 0 & 0 & 0 \\ -80 & 0 & 0 & -80 & 0 & 0 & 1 & 0 & 0 \\ 0 & -560 & 0 & 0 & -140 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2240 & 0 & 0 & -224 & 0 & 0 & 1 \\ -17920 & 0 & 0 & -6720 & 0 & 0 & -336 & 0 & 0 \end{pmatrix}.$$

The A and Z generating functions of P are determined below.

We know that for an exponential Riordan array $[g, f]$

$$\mathrm{A}(z) = \left(1 - z^3\right)^{2/3} \text{ and } \bar{f}(z) = \mathrm{sm}^{-1}(z)$$

Now,

$$Z(z) = \frac{g'(\bar{f})}{g(\bar{f})}$$

where

$$g(z) = \mathrm{sm}'(z) = \mathrm{cm}^2(z)$$

So

$$g(\bar{f}) = \mathrm{cm}^2(\mathrm{sm}^{-1}(z))$$

But

$$\mathrm{cm}^3 + \mathrm{sm}^3 = 1 \implies \mathrm{cm} = \left(1 - \mathrm{sm}^3\right)^{1/3}$$

So

$$\begin{aligned}
\mathrm{cm}^2(\mathrm{sm}^{-1}(z)) &= \left[(1 - \mathrm{sm}^3(\mathrm{sm}^{-1}(z))^{1/3}\right]^2 \\
&= \left(1 - z^3\right)^{2/3} \\
&= g(\bar{f})
\end{aligned}$$

Also

$$\begin{aligned}
g'(\bar{f}) &= (\mathrm{cm}^2)'(\mathrm{sm}^{-1}(z)) \\
&= 2(\mathrm{cm} \cdot \mathrm{cm}')(\mathrm{sm}^{-1}(z)) \\
&= 2\mathrm{cm}(\mathrm{sm}^{-1}(z))(-\mathrm{sm}^2(\mathrm{sm}^{-1}(z))) \\
&= 2(1 - \mathrm{sm}^3(\mathrm{sm}^{-1}(z))^{1/3}(-z^2) \\
&= 2(1 - z^3)^{1/3}(-z^2)
\end{aligned}$$

$$\implies \frac{g'(\bar{f})}{g(\bar{f})} = \frac{-2z^2(1 - z^3)^{1/3}}{(1 - z^3)^{2/3}} = \frac{-2z^2}{(1 - z^3)^{1/3}}.$$

$$\text{That is} \quad \mathrm{Z}(z) = \frac{-2z^2}{(1 - z^3)^{1/3}} \quad \text{and} \quad \mathrm{A}(z) = (1 - z^3)^{2/3}.$$

4.2.2 The Riordan array $\left[\frac{d}{dz} z\mathbf{cm}(z), z\mathbf{cm}(z)\right]$

Consider the coefficient array of $\left[\frac{d}{dz} z\mathrm{cm}(z), z\mathrm{cm}(z)\right]$

$$
\mathrm{M} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -40 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -120 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
280 & 0 & 0 & -280 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 4480 & 0 & 0 & -560 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 30240 & 0 & 0 & -1008 & 0 & 0 & 1 & 0 & 0 \\
-36800 & 0 & 0 & 134400 & 0 & 0 & -1680 & 0 & 0 & 1 & 0 \\
0 & -1144000 & 0 & 0 & 462000 & 0 & 0 & -2640 & 0 & 0 & 1
\end{pmatrix}.
$$

The production matrix M is given by:

$$
\mathrm{P} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -32 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -80 & 0 & 0 & 1 & 0 & 0 & 0 \\
-680 & 0 & 0 & -160 & 0 & 0 & 1 & 0 & 0 \\
0 & -4760 & 0 & 0 & -280 & 0 & 0 & 1 & 0 \\
0 & 0 & -19040 & 0 & 0 & -448 & 0 & 0 & 1 \\
-480320 & 0 & 0 & -57120 & 0 & 0 & -672 & 0 & 0
\end{pmatrix}.
$$

4.2.3 The Riordan array $[1, \mathrm{sm}(z)]$

The proper exponential Riordan array $[1, \mathrm{sm}(z)]$ of the Lagrange subgroup has the coefficient matrix given by

$$
\mathrm{M} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -20 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -60 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 160 & 0 & 0 & -140 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1840 & 0 & 0 & -280 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10800 & 0 & 0 & -504 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -20800 & 0 & 0 & 44400 & 0 & 0 & -840 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -440000 & 0 & 0 & 145200 & 0 & 0 & -1320 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -4276800 & 0 & 0 & 403920 & 0 & 0 & -1980 & 0 & 0 & 1
\end{pmatrix}.
$$

The production matrix corresponding to the coefficient matrix of M is given by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -16 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -40 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -80 & 0 & 0 & -80 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -560 & 0 & 0 & -140 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2240 & 0 & 0 & -224 & 0 & 0 & 1 \\ 0 & -17920 & 0 & 0 & -6720 & 0 & 0 & -336 & 0 & 0 \end{pmatrix}.$$

The A generating function corresponding to the production matrix P is given by $A(z) = (1 - z^3)^{2/3}$.

4.2.4 The Riordan array $[\mathbf{cm}(z), z]$

The proper exponential Riordan array $[\mathrm{cm}(z), z]$ of the Appell subgroup has the coefficient matrix given by:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & 0 & 0 & -40 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 280 & 0 & 0 & -70 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1120 & 0 & 0 & -112 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -3680 & 0 & 0 & 3360 & 0 & 0 & -168 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -36800 & 0 & 0 & 8400 & 0 & 0 & -240 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -202400 & 0 & 0 & 18480 & 0 & 0 & -330 & 0 & 0 & 1 & 0 \\ 880000 & 0 & 0 & -809600 & 0 & 0 & 36960 & 0 & 0 & -440 & 0 & 0 & 1 \end{pmatrix}.$$

The production matrix corresponding to the coefficient matrix of M is given by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -30 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -42 & 0 & 0 & 1 & 0 \\ -1440 & 0 & 0 & 0 & 0 & 0 & -56 & 0 & 0 & 1 \\ 0 & -12960 & 0 & 0 & 0 & 0 & 0 & -72 & 0 & 0 \end{pmatrix}.$$

Let $[g(z), f(z)] = [\mathrm{cm}(z), z]$ .Then the A generating function of P is given by by $\mathrm{A}(z) = 1$ since $f'(z) = 1$. The Z generating function is

$$\frac{g'(\bar{f})}{g(\bar{f})} = \frac{\mathrm{cm}'(z)}{\mathrm{cm}(z)}$$
$$= \frac{d}{dz}\ln(\mathrm{cm}(z)).$$

Chapter 5

Embedded Sub-Matrices from a Riordan array

5.1 Introduction

Embedded sub-structures forming new Riordan arrays from existing Riordan arrays have been investigated from different contexts. The various research to date on embedded Riordan arrays are briefly presented below.

- Barry [12] investigates the concept of embedded Riordan arrays. In the case of embedded Riordan arrays, the process of obtaining the embedded Riordan array from an existing Riordan array $(d(t), h(t))$ involves taking every second column after the first column resulting in another lower triangular matrix A that can be represented as $A = \left(d, \frac{h^2}{z}\right)$.

- The m complementary Riordan arrays is presented in the publication by Luzón et al. [69]. The concept of complementary Riordan arrays states that if $D = \mathcal{R}\left(d(t), h(t)\right)$ then its [m] complementary array is the Riordan array.
$$D^{[m]} = \left(d(\bar{h}(t))\bar{h}'(t)\left(\frac{t}{\bar{h}(t)}\right)^{m+1}, \bar{h}(t)\right).$$

- In the r-shifted central coefficients [126], by setting $D = (d(z), h(z)) = (d_{i,j})_{i,j\geq 0}$ to represent a Riordan array then the r-shifted central coefficients results to the new Riordan array $D_{2n+r,n+r,}$ where $n, r \in \mathbb{N}_0$.

- The row elimination procedure by Brietzke[19], is a type of embedded Riordan array constructed from a given Riordan array uses a process of eliminating entire rows and some parts of the remaining rows. That is if given a proper Riordan array $\{d_{n,k}\}_{n,k\geq 0}$, then for any integers $p \geq 2$ and $r \geq 0$, $\tilde{d}_{n,k} = d_{pn+r,(p-1)n+r+k}(n, k \geq 0)$ defines a new Riordan array.

All these mentioned methods have relied on manipulating the existing generating functions or by shifting the position of the elements that make up the original ordinary Riordan array. In this chapter we extend the existing examples of embedded Riordan arrays by identifying some sub-matrices forming new Riordan arrays from an existing exponential Riordan array generated from elliptic Jacobi functions. Based on the structure of the elements of Jacobi Riordan arrays and inspired by the existing knowledge on embedded Riordan arrays, some interesting sub-matrices corresponding to the monic polynomials from elliptic Jacobi *dc* function are examined. In addition, to using the already known methods of determining embedded Riordan arrays, two new recent techniques to form new Riordan arrays from existing ones are presented using examples of some elliptic Jacobi functions. These are the r-shifted central triangles of a Riordan array connected to the r-shifted central coefficients and the one-parameter family of Riordan arrays derived from a given Riordan array connected to the complementary Riordan arrays. The examples presented below using Jacobi elliptic functions illustrate the application of some of these techniques.

5.2 The submatrices of Riordan arrays generated by Elliptic Functions

In this section we investigate the submatrices derived from the first or second column of some Riordan arrays generated from elliptic functions. The case of the sub-matrices from the first column are associated to their original Riordan matrices belonging to the Appell subgroup. On the other hand the submatrices arising from the second column are associated to their original Riordan matrices belonging to the Lagrange subgroup.

5.2.1 The submatrix of $\left[\frac{\mathbf{dn}(z,m)}{\mathbf{cn}(z,m)}, z\right]$

The proper exponential Riordan array $\left[\frac{\mathrm{dn}(z,m)}{\mathrm{cn}(z,m)}, z\right]$ has coefficient matrix given by

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1-m & 0 & 1 & 0 & 0 & 0 \\ 0 & 3-3m & 0 & 1 & 0 & 0 \\ m^2-6m+5 & 0 & 6-6m & 0 & 1 & 0 \\ 0 & 5\left(m^2-6m+5\right) & 0 & -10(m-1) & 0 & 1 \end{pmatrix}.$$

The non-zero elements of the first column of matrix C are associated to the matrix:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 5 & -6 & 1 & 0 & 0 & 0 \\ 61 & -107 & 47 & -1 & 0 & 0 \\ 1385 & -3116 & 2142 & -412 & 1 & 0 \\ 50521 & -138933 & 130250 & -45530 & 3693 & -1 \end{pmatrix}.$$

The first column of D forms the sequence $(1, 1, 5, 61, 1385, ...)$ which are referred to as the Euler numbers having e.g.f $\sec(z)$ corresponding to **A000364.** The row sums of D are $\{1, 0, 0, 0, 0, 0\}$ having the simple generating function z corresponding to **A010054** .

Alternatively, the matrix D can be transformed by multiplying with $(-1)^n$ where n is the column number resulting to

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 5 & 6 & 1 & 0 & 0 & 0 \\ 61 & 107 & 47 & 1 & 0 & 0 \\ 1385 & 3116 & 2142 & 412 & 1 & 0 \\ 50521 & 138933 & 130250 & 45530 & 3693 & 1 \end{pmatrix}.$$

In particular, the row sums of E form the sequence $(1, 2, 12, 216, 7056, 368928, ...)$ corresponding to **A153302** with g.f $A(z) = \mathrm{cm4}(z)^2 + \mathrm{sm4}(z)^2$ where $\mathrm{cm4}(z)$ and $\mathrm{sm4}(z)$ are the g.f.s of **A153300** and **A153301**, respectively, that satisfy $\mathrm{cm4}(z)^4 - \mathrm{sm4}(z)^4 = 1$.

5.2.2 The submatrix of $\left[\frac{\mathbf{dn}(z,m)}{\mathbf{cn}(z,m)^2}, z\right]$

The coefficient matrix of the proper exponential Riordan array $\left[\frac{\mathrm{dn}(z,m)}{\mathrm{cn}(z,m)^2}, z\right]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2-m & 0 & 1 & 0 & 0 & 0 \\ 0 & 6-3m & 0 & 1 & 0 & 0 \\ m^2-16m+16 & 0 & 12-6m & 0 & 1 & 0 \\ 0 & 5m^2-80m+80 & 0 & 20-10m & 0 & 1 \end{pmatrix}$$

The non-zero elements of the first column of the matrix A are associated to the matrix:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 16 & -16 & 1 & 0 & 0 & 0 & 0 \\ 272 & -408 & 138 & -1 & 0 & 0 & 0 \\ 7936 & -15872 & 9168 & -1232 & 1 & 0 & 0 \\ 353792 & -884480 & 729728 & -210112 & 11074 & -1 & 0 \\ 22368256 & -67104768 & 71997696 & -32154112 & 4992576 & -99648 & 1 \end{pmatrix}.$$

The elements of the first column of B forms the sequence $(1, 2, 16, 272, ..)$ having E.g.f $\tan(z)$ corresponding to **A000182**. Furthermore, the row sums of B are given by $(1, 1, 1, 1, 1, 1, 1)$ having e.g.f e^z which corresponds to **A000012**.

Using the generating function for the row sums of B given by e^z and the generating function of the first column of B given by $\tan(z)$ we determine a new Riordan array from these results below. Recall that for an exponential Riordan array $[g(z), f(z)]$, the row sums are given by

$$g(z)e^{f(z)}$$

where $g(z) \in \mathcal{F}_0$ and $f(z) \in \mathcal{F}_1$.

But $\tan(z) \in \mathcal{F}_1$ since

$$\tan(z) = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + O\left(z^9\right).$$

Therefore solving to determine the first generating function of the Riordam array we get

$$\begin{aligned} e^z &= g(z)e^{\tan(z)} \\ g(z) &= \frac{e^z}{e^{\tan(z)}} \\ &= e^{z-\tan(z)}. \end{aligned}$$

That is $g(z) = e^{z-\tan(z)} \in \mathcal{F}_0$ since

$$e^{z-\tan(z)} = 1 - \frac{z^3}{3} - \frac{2z^5}{15} + \frac{z^6}{18} - \frac{17z^7}{315} + \frac{2z^8}{45} + O\left(z^9\right).$$

The coefficient matrix of the exponential Riordan array

$$\left[e^{z-\tan(z)}, \tan(z)\right]$$

is given by

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -8 & 8 & 0 & 1 & 0 & 0 \\ -16 & 16 & -20 & 20 & 0 & 1 & 0 \\ 40 & -176 & 136 & -40 & 40 & 0 & 1 \end{pmatrix}.$$

The production matrix of C is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & -6 & 6 & 0 & 1 & 0 \\ 0 & 0 & -12 & 12 & 0 & 1 \\ 0 & 0 & 0 & -20 & 20 & 0 \end{pmatrix}.$$

5.2.3 The submatrix of $\left[\frac{\mathbf{dn}(z,m)}{\mathbf{cn}(z,m)^3}, z\right]$

The coefficient matrix of the proper exponential Riordan array $\left[\frac{\mathrm{dn}(z,m)}{\mathrm{cn}(z,m)^3}, z\right]$ is given by

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 3-m & 0 & 1 & 0 & 0 & 0 \\ 0 & 9-3m & 0 & 1 & 0 & 0 \\ m^2-26m+33 & 0 & -6(m-3) & 0 & 1 & 0 \\ 0 & 5\left(m^2-26m+33\right) & 0 & -10(m-3) & 0 & 1 \end{pmatrix}.$$

The non-zero elements of the first column of matrix F are associated to the matrix:

$$G=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 & 0 \\ 33 & -26 & 1 & 0 & 0 & 0 \\ 723 & -919 & 229 & -1 & 0 & 0 \\ 25953 & -45764 & 21990 & -2052 & 1 & 0 \\ 1376643 & -3110077 & 2240006 & -524514 & 18455 & -1 \end{pmatrix}.$$

The row sums of G form the sequence $(1,2,8,32,128,512,...)$ which corresponds to **A081294** having e.g.f $e^{2z}\cosh(2z)$.

5.2.4 The submatrix of $\left[\frac{\mathbf{dn}(z,m)}{\mathbf{cn}(z,m)^4}, z\right]$

The coefficient matrix of the proper exponential Riordan array $\left[\frac{\mathrm{dn}(z,m)}{\mathrm{cn}(z,m)^4}, z\right]$ is given by

$$H=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 4-m & 0 & 1 & 0 & 0 & 0 \\ 0 & -3(m-4) & 0 & 1 & 0 & 0 \\ m^2-36m+56 & 0 & -6(m-4) & 0 & 1 & 0 \\ 0 & 5\left(m^2-36m+56\right) & 0 & -10(m-4) & 0 & 1 \end{pmatrix}.$$

The non-zero elements of the first column of matrix H are associated to the matrix:

$$I=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 & 0 & 0 \\ 56 & -36 & 1 & 0 & 0 & 0 \\ 1504 & -1640 & 320 & -1 & 0 & 0 \\ 64256 & -100352 & 40608 & -2872 & 1 & 0 \\ 3963904 & -8104704 & 5118464 & -988736 & 25836 & -1 \end{pmatrix}.$$

The row sums of I form the sequence $(1,3,21,183,1641,14763,...)$ which corresponds to **A054879** having the e.g.f $\cosh^3(z)$.

5.2.5 The submatrix of $\left[\frac{\mathbf{dn}(z,m)}{\mathbf{cn}(z,m)^5}, z\right]$

The coefficient matrix of the proper exponential Riordan array $\left[\frac{\mathrm{dn}(z,m)}{\mathrm{cn}(z,m)^5}, z\right]$ is given by

$$J=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 5-m & 0 & 1 & 0 & 0 & 0 \\ 0 & -3(m-5) & 0 & 1 & 0 & 0 \\ m^2-46m+85 & 0 & -6(m-5) & 0 & 1 & 0 \\ 0 & 5\left(m^2-46m+85\right) & 0 & -10(m-5) & 0 & 1 \end{pmatrix}.$$

The non-zero elements of the first column of matrix J is associated to the matrix

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & -1 & 0 & 0 & 0 & 0 \\ 85 & -46 & 1 & 0 & 0 & 0 \\ 2705 & -2571 & 411 & -1 & 0 & 0 \\ 134185 & -187196 & 65022 & -3692 & 1 & 0 \\ 9451805 & -17573141 & 9822482 & -1602778 & 33217 & -1 \end{pmatrix}.$$

The row sums of K form the sequence $(1, 4, 40, 544, 8320, 131584, ...)$ which corresponds to **A092812** having the e.g.f $\cosh^4(z)$.

5.2.6 The submatrix of $\left[\frac{\mathbf{dn}(z,m)}{\mathbf{cn}(z,m)^6}, z\right]$

The coefficient matrix of the proper exponential Riordan array $\left[\frac{\mathrm{dn}(z,m)}{\mathrm{cn}(z,m)^6}, z\right]$ is given by

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 6-m & 0 & 1 & 0 & 0 & 0 \\ 0 & -3(m-6) & 0 & 1 & 0 & 0 \\ m^2-56m+120 & 0 & -6(m-6) & 0 & 1 & 0 \\ 0 & 5\left(m^2-56m+120\right) & 0 & -10(m-6) & 0 & 1 \end{pmatrix}.$$

The non-zero elements of the first column of the matrix M is associated to the matrix:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & -1 & 0 & 0 & 0 & 0 & 0 \\ 120 & -56 & 1 & 0 & 0 & 0 & 0 \\ 4416 & -3712 & 502 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

The row sums of M form the sequence $(1, 5, 65, 1205, 26465, 628805, ...)$ which corresponds to **A121822** having the e.g.f $\cosh^5(z)$.

5.2.7 The submatrix of $[\mathbf{cn}(z,m)^2, \mathbf{sn}(z,m)]$

The coefficient matrix of $\left[\mathrm{cn}(z,m)^2, \mathrm{sn}(z,m)\right]$ is given by

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -m-7 & 0 & 1 & 0 & 0 \\ 8(m+1) & 0 & -4(m+4) & 0 & 1 & 0 \\ 0 & m^2+74m+61 & 0 & -10(m+3) & 0 & 1 \end{pmatrix}.$$

The non-zero elements of the second column of L are associated to the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & -1 & 0 & 0 & 0 & 0 & 0 \\ 61 & 74 & 1 & 0 & 0 & 0 & 0 \\ -547 & -2739 & -681 & -1 & 0 & 0 & 0 \\ 4921 & 80788 & 85038 & 6148 & 1 & 0 & 0 \\ -44287 & -2169797 & -6590134 & -2324554 & -55355 & -1 & 0 \\ 398581 & 55949982 & 413000631 & 421686548 & 60344691 & 498222 & 1 \end{pmatrix}.$$

The matrix M can be transformed by multiplying with $(-1)^{n+1}$ where n is the row number s.t. $n = 1, 2, 3, ...$ to obtain the matrix

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 61 & 74 & 1 & 0 & 0 & 0 & 0 \\ 547 & 2739 & 681 & 1 & 0 & 0 & 0 \\ 4921 & 80788 & 85038 & 6148 & 1 & 0 & 0 \\ 44287 & 2169797 & 6590134 & 2324554 & 55355 & 1 & 0 \\ 398581 & 55949982 & 413000631 & 421686548 & 60344691 & 498222 & 1 \end{pmatrix}.$$

The elements of the first column of N forms the sequence $(1, 7, 61, 547, 4921, 44287, ...)$ which corresponds to **A066443** having e.g.f $3e^{9z} + \frac{e^z}{4}$. The row sums of N form the sequence $(1, 8, 136, 3968, 176896, 11184128, 951878656, ...)$ corresponding to the non zero elements of **A024283** having e.g.f $\frac{1}{2}\tan(z)^2$.

5.2.8 The submatrix of $\left[1, \frac{\mathbf{sn}(z,m)}{1+\mathbf{sn}(z,m)^2}\right]$

The coefficient matrix of $\left[1, \frac{\text{sn}(z,m)}{1+\text{sn}(z,m)^2}\right]$ is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -m-7 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4(m+7) & 0 & 1 & 0 \\ 0 & m^2+74m+181 & 0 & -10(m+7) & 0 & 1 \end{pmatrix}.$$

The second column of P is generated from the expansion of $\frac{\text{sn}(z,m)}{1+\text{sn}(z,m)^2}$ The non-zero elements of the second column of P are associated with the matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & -1 & 0 & 0 & 0 & 0 & 0 \\ 181 & 74 & 1 & 0 & 0 & 0 & 0 \\ -9787 & -6939 & -681 & -1 & 0 & 0 & 0 \\ 907081 & 917428 & 200958 & 6148 & 1 & 0 & 0 \\ -128445967 & -168735317 & -59725414 & -5320954 & -55355 & -1 & 0 \\ 25794366781 & 41682334782 & 20629917351 & 3377119028 & 136140411 & 498222 & 1 \end{pmatrix}.$$

The matrix Q can be transformed by multiplying with $(-1)^{n+1}$ where n is the row number s.t. $n = 1, 2, 3, ...$ to obtain the matrix

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 181 & 74 & 1 & 0 & 0 & 0 & 0 \\ 9787 & 6939 & 681 & 1 & 0 & 0 & 0 \\ 907081 & 917428 & 200958 & 6148 & 1 & 0 & 0 \\ 128445967 & 168735317 & 59725414 & 5320954 & 55355 & 1 & 0 \\ 25794366781 & 41682334782 & 20629917351 & 3377119028 & 136140411 & 498222 & 1 \end{pmatrix}.$$

The row sums of R form the sequence $(1, 8, 256, 17408, 2031616, 362283008, 91620376576, ...)$ which corresponds to **A253165** known as the **generalized Riemann zeta function** $(-1)^n 2^{6n+3} \left(\left(2^{-2n-1} - 1 \right) \zeta(-2n-1) - \zeta(-2n-1) \right)$. The sequence **A253165** can alternatively be generated from $(2n!)[z^{2n}] \sec(2z)^2$, or in other words, $\operatorname{sech}(2z)^2$ is the g.f of the aerated sequence.

5.3 Embedded Riordan arrays from Riordan arrays generated by Elliptic Functions

In this section we apply the technique of forming a new Riordan array using the formula $\left[d(z), \frac{h^2(z)}{z}\right]$ from the original Riordan array $[d(z), h(z)]$, briefly introduced in (5.1).

5.3.1 The embedded Riordan array of $[\mathbf{cn}(z, m), \mathbf{sn}(z, m)]$

The Riordan array $[\operatorname{cn}(z, m), \operatorname{sn}(z, m)]$ produces the embedded Riordan array $A = \left[\operatorname{cn}(z, m), \frac{\operatorname{sn}(z,m)^2}{z}\right]$. The coefficient matrix corresponding to A is given by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -2m-5 & 0 & 1 & 0 \\ 4m+1 & 0 & -2(4m+7) & 0 & 1 \end{pmatrix}.$$

The matrix B is associated to the production matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -2(m+2) & 0 & 1 \\ 2(m-2) & 0 & -6m-9 & 0 \end{pmatrix}.$$

5.3.2 The embedded Riordan array of $[\mathbf{dn}(z,m), \mathbf{sn}(z,m)]$

The Riordan array $[\mathrm{dn}(z,m), \mathrm{sn}(z,m)]$ produces the embedded Riordan array $A = \left[\mathrm{dn}(z,m), \frac{\mathrm{sn}(z,m)^2}{z}\right]$. The coefficient matrix corresponding to A is given by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -m & 0 & 1 & 0 & 0 \\ 0 & -5m-2 & 0 & 1 & 0 \\ m(m+4) & 0 & -2(7m+4) & 0 & 1 \end{pmatrix}.$$

The matrix B is associated to the production matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -m & 0 & 1 & 0 \\ 0 & -4m-2 & 0 & 1 \\ 2(1-2m)m & 0 & -9m-6 & 0 \end{pmatrix}.$$

5.3.3 The embedded Riordan array of $[\mathbf{cn}(z,m), z\mathbf{cn}(z,m)]$

The Riordan array $[\mathrm{cn}(z,m), z\mathrm{cn}(z,m)]$ produces the embedded Riordan array $A = \left[\mathrm{cn}(z,m), \frac{(z\mathrm{cn}(z,m))^2}{z}\right] \equiv \left[\mathrm{cn}(z,m), z\mathrm{cn}(z,m)^2\right]$. The coefficient matrix corresponding to A is given by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -9 & 0 & 1 & 0 \\ 4m+1 & 0 & -30 & 0 & 1 \end{pmatrix}.$$

The matrix B is associated to the production matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -8 & 0 & 1 \\ 4m-8 & 0 & -21 & 0 \end{pmatrix}.$$

5.3.4 The embedded Riordan array of $[\mathbf{dn}(z,m), z\mathbf{dn}(z,m)]$

The Riordan array $A = [\mathrm{dn}(z,m), z\mathrm{dn}(z,m)]$ produces the embedded Riordan array $B = \left[\mathrm{dn}(z,m), \frac{(z\mathrm{dn}(z,m))^2}{z}\right] \equiv \left[\mathrm{dn}(z,m), z\mathrm{dn}(z,m)^2\right]$. The coefficient matrix corresponding to A is given by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -m & 0 & 1 & 0 & 0 \\ 0 & -9m & 0 & 1 & 0 \\ m^2+4m & 0 & -30m & 0 & 1 \end{pmatrix}.$$

The matrix B is associated to the production matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -m & 0 & 1 & 0 \\ 0 & -8m & 0 & 1 \\ 4(1-2m)m & 0 & -21m & 0 \end{pmatrix}.$$

5.4 Riordan arrays derived from the r-shifted central triangles of a given Riordan array

The family of r-shifted central coefficients $a_{2n+r,n+r}$ of a Riordan array were originally introduced by the authors Zheng and Yang [126]. A family of matrices $c(A;r)$ known as the r-shifted central triangles of the Riordan array A having general term $a_{2n+r,n+k+r}$ can be constructed from the original Riordan array A. These new matrices arising from A have the r-shifted central coefficients as their leftmost column with all $1'$s on their principal diagonal. The outcome of trying to characterize the family of matrices $c(A;r)$ results in a one parameter family of Riordan arrays. These results for $c(A;r)$ when $r=1$ are summarized below.

Theorem 5.4.1 *The shifted central triangle $c(A;1)$ of the Riordan array $A = [g(z), zf(z)]$ is a Riordan array which admits the following factorization.*

$$c(A;1) = \left[\frac{\phi'}{f(\phi)}, \phi\right] \cdot A,$$

where

$$\phi(z) = Rev\left(\frac{z}{f(z)}\right) \text{ and } \phi'(z) = \frac{d}{dz}\phi(z).$$

Corollary 5.4.2 *We have*

$$c(A;1) = \left[\phi' \frac{g(\phi)}{f(\phi)}, \phi f(\phi)\right].$$

Corollary 5.4.3 *We have*

$$c(A;1) = \left[\frac{f(z)}{\phi'(\frac{z}{f(z)})}, \frac{z}{f(z)}\right]^{-1} \cdot [g(z), zf(z)].$$

Corollary 5.4.4

$$c(A;1)^{-1} = \left[\frac{1}{g(\bar{v})} \frac{f(\bar{v})}{\phi'\left(\frac{\bar{v}}{f(\bar{v})}\right)}, \frac{\bar{v}}{f(\bar{v})} \right],$$

where $v(z) = zf(z)$.

Corollary 5.4.5 *Let* $A = [f(z), zf(z)]$ *be a member of the Bell subgroup of the Riordan group. Then*

$$c(A;1) = [\phi', \phi f(\phi)],$$

and

$$c(A;1)^{-1} = \left[\frac{1}{\phi'\left(\frac{\bar{v}}{f(\bar{v})}\right)}, \frac{\bar{v}}{f(\bar{v})} \right].$$

We investigate some examples corresponding to each of these results below.

5.4.1 $\left[\mathbf{cn}(z,m), \frac{z^2}{\mathbf{sn}(z,m)}\right]$

We verify the results of Corollary (5.4.3) for the case of the the Riordan array

$$A = \left[\mathrm{cn}(z,m), \frac{z^2}{\mathrm{sn}(z,m)}\right] \equiv \left[\mathrm{cn}(z,m), z^2\mathrm{ns}(z,m)\right].$$

The Riordan array A produces the Riordan array

$$B = \left[\frac{-z\left(m\mathrm{sn}(z|m)^2 - 1\right)\mathrm{cd}\left(\mathrm{sn}^{-1}(\mathrm{sn}(z|m)|m)\right| m\right)}{\mathrm{sn}(z|m)}, \mathrm{sn}(z|m) \right]$$

by applying the formula $\left[\frac{f(z)}{\phi'(\frac{z}{f(z)})}, \frac{z}{f(z)}\right]$ where

$$f(z) = \frac{z}{\mathrm{sn}(z,m)} \text{ and } \phi(z) = \mathrm{sn}^{-1}(z,m).$$

The coefficient matrix of B is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{2}{3}(m+1) & 0 & 1 & 0 & 0 & 0 \\ 0 & -3(m+1) & 0 & 1 & 0 & 0 \\ -\frac{8}{15}((m-16)m+1) & 0 & -8(m+1) & 0 & 1 & 0 \\ 0 & 5(m(m+14)+1) & 0 & -\frac{50}{3}(m+1) & 0 & 1 \end{pmatrix}.$$

5.5 A one-parameter family of Riordan arrays derived from a given Riordan array

For every integer s a transformation of a Riordan array can be defined which yields another Riordan array. By setting a value for the integer s the effects of these transformations on some Jacobi Riordan arrays are evaluated for new combinatorial results arising from the new Riordan matrices.

5.5.1 Examples using the inverse transformation $T^{(s)^{-1}}$

5.5.1.1 The case of $[g, f] = [\mathbf{cd}(z, m), z]$

We apply the formula $T^{(s)^{-1}} = [g, f]^{-1} \cdot \left[\psi'(z)\left(\frac{\psi}{z}\right)^{s-1}, \psi\right]$ where $\psi(z) = \frac{z}{1+f(z)}$, to the original Riordan array $[g, f] = [cd(z, m), z]$. In particular, if $s = 2$ we have that

$$\psi = \frac{z}{1+z} \implies \left[\psi'(z)\left(\frac{\psi}{z}\right)^{s-1}, \psi\right] = \left[\frac{1}{(1+z)^3}, \frac{z}{1+z}\right].$$

$$T^{(2)^{-1}} = \left[\frac{1}{(1+z)^3 cd(z, m)}, \frac{z}{1+z}\right].$$

The matrix representing $T^{(2)^{-1}}$ is given by

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 & 0 \\ 13-m & -8 & 1 & 0 & 0 & 0 \\ 9m-69 & 63-3m & -15 & 1 & 0 & 0 \\ (m-78)m+437 & 48(m-11) & -6(m-31) & -24 & 1 & 0 \\ -15((m-46)m+213) & 5((m-126)m+965) & 150(m-15) & -10(m-43) & -35 & 1 \end{pmatrix}.$$

The production matrix of T is given by

$$T_p = \begin{pmatrix} -3 & 1 & 0 & 0 \\ 4-m & -5 & 1 & 0 \\ 2-2m & -2(m-5) & -7 & 1 \\ -2(m-1)(m+4) & 6-6m & -3(m-6) & -9 \end{pmatrix}.$$

Remark 1: If $m = 1$ The coefficient matrix of T becomes

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 12 & -8 & 1 & 0 & 0 & 0 & 0 \\ -60 & 60 & -15 & 1 & 0 & 0 & 0 \\ 360 & -480 & 180 & -24 & 1 & 0 & 0 \\ -2520 & 4200 & -2100 & 420 & -35 & 1 & 0 \\ 20160 & -40320 & 25200 & -6720 & 840 & -48 & 1 \end{pmatrix}.$$

The coefficient matrix $T_1 = \left[\frac{1}{(1+z)^3}, \frac{z}{1+z}\right] \equiv \left[\psi'(z)\left(\frac{\psi}{z}\right)^{s-1}, \psi\right]$.

If we multiply T_1 by $(-1)^{n+1}$ where n represents the row number of T_1 starting from 1, we get the matrix

$$T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 12 & -8 & 1 & 0 & 0 & 0 & 0 \\ 60 & -60 & 15 & -1 & 0 & 0 & 0 \\ 360 & -480 & 180 & -24 & 1 & 0 & 0 \\ 2520 & -4200 & 2100 & -420 & 35 & -1 & 0 \\ 20160 & -40320 & 25200 & -6720 & 840 & -48 & 1 \end{pmatrix}.$$

The matrix T_2 corresponds to **A062139** which is the coefficient triangle of the generalized Laguerre polynomials $n! * L(n, 2, z)$ for rising powers of z. The row sums of the signed triangle T_2 are $(1, 2, 5, 14, 37, 34, -887, ...)$ corresponding to **A062197** with e.g.f $\frac{e^{\frac{-z}{1-z}}}{(1-z)^3}$.

Remark 2: The exponential Riordan array $\left[\frac{1}{(1+z)^3}, \frac{z}{1+z}\right]$ arising from the transformation $T^{(s)^{-1}}$ has a tri-diagonal production matrix for $m = 1$ given by

$$B = \begin{pmatrix} -3 & 1 & 0 & 0 & 0 \\ 3 & -5 & 1 & 0 & 0 \\ 0 & 8 & -7 & 1 & 0 \\ 0 & 0 & 15 & -9 & 1 \\ 0 & 0 & 0 & 24 & -11 \end{pmatrix}.$$

This indicates that the inverse matrix of $\left[\frac{1}{(1+z)^3}, \frac{z}{1+z}\right]$ is associated to the coefficients of a family of orthogonal polynomial sequences.

Remark 3: The r and c generating function of T_1 are respectively

$$r(z) = 1 - z^2 \qquad c(z) = 3 - 3z.$$

5.5.1.2 The case of $[g, f] = [\mathbf{cd}(z, m), \mathbf{sn}(z, m)]$

We apply the formula $T^{(s)^{-1}} = [g, f]^{-1} \cdot \left[\psi'(z)\left(\frac{\psi}{z}\right)^{s-1}, \psi\right]$ where $\psi(z) = \frac{z}{1+f(z)}$, to the original Riordan array $[g, f] = [\mathrm{cd}(z, m), \mathrm{sn}(z, m)]$. In particular, if $s = 2$ we have that

$$\psi = \frac{z}{1 + \mathrm{sn}(z, m)} \implies \left[\psi'(z)\left(\frac{\psi}{z}\right)^{s-1}, \psi\right] =$$

$$\left[\frac{-z\mathrm{cn}(z|m)\mathrm{dn}(z|m) + \mathrm{sn}(z|m) + 1}{(\mathrm{sn}(z|m) + 1)^3}, \frac{z}{1 + sn(z, m)}\right].$$

$$[g, f]^{-1} = \left[\frac{\left(mz^2 - 1\right)\mathrm{cd}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right)}{z^2 - 1}, sn^{-1}(z, m)\right]$$

$T^{(2)^{-1}}$ equals to

$$\left[\frac{\mathrm{cd}\left(\mathrm{sn}^{-1}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right)\middle|\, m\left(z(-\mathrm{cn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right))\mathrm{dn}\left(\mathrm{sn}^{-1}(z|m)\middle|\, m\right) + z + 1\right)\right.}{(z + 1)^3}\right.$$

$$\left., \frac{z}{1 + sn(z, m)}\right].$$

Chapter 6

Riordan arrays and Solutions to Differential Equations

6.1 The Sturm-Liouville equation

The **Sturm-Liouville equation** [28] is a real second order differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = -\lambda w(x)y. \tag{6.1}$$

The equation (6.1) is derived under general conditions by rewriting a given second order homogenous differential equation in one dimension of the form

$$a(x)\frac{d^2}{dx^2}y(x) + b(x)\frac{d}{dx}y(x) + c(x)y(x) = 0,$$

in another form of differential equation involving a self-adjoint under suitable boundary conditions [86]. In this sense, the Sturm-Liouville operator can be defined by

$$Ly = 0$$

where

$$L = \frac{d}{dx}p(x)\frac{d}{dx} - q(x).$$

The eigen-value equation associated with L can be written as

$$Ly + \lambda wy = 0 \equiv \frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] - q(x)y(x) + \lambda w(x)y(x) = 0,$$

where λ is the eigen value corresponding to the eigen function $y(x)$ satisfying the boundary conditions, the real valued function $\omega(x) > 0$ is the weight function.

A catalogue providing details of over 50 examples of Sturm-Liouville differential equations has been presented by Everitt [38]. Most of these differential equations are directly related to problems in mathematical physics. Riordan arrays can represent the solution to systems of differential equations. The commonest examples come from Sturm Liouville systems with parameter λ that depends on n. In the account that follows a connection between the solutions to some Sturm-Liouville differential equations with ordinary and exponential Riordan arrays will be outlined. For the purpose of this investigation the case of the Laguerre equation, Bessel equation, Hermite equation, Morgan-Voyce equation and Chebyshev equation are considered. Each of these equations will be treated in such a way as to show that the coefficient arrays corresponding to the polynomial sequences that make up their solutions are Riordan arrays.

6.1.1 Laguerre Polynomials

The general form of the Laguerre differential equation [121] is

$$xy'' + (1 - x)y' + \lambda y = 0. \tag{6.2}$$

In addition, (6.2) is considered a special case of the general associated or generalized Laguerre differential equation defined by

$$xy'' + (v + 1 - x)y' + \lambda y = 0 \tag{6.3}$$

where λ and v are real numbers. The solutions to the Laguerre differential equation (6.2) are given by a polynomial sequence known as **Laguerre polynomials** which are most often denoted as $L_0, L_1, L_2, ...$ and can be expressed explicitly by the formula

$$L_n(x) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k.$$

The generating function of the Laguerre polynomials is given by

$$\frac{e^{-\left(\frac{xt}{1-t}\right)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!}.$$

The Laguerre polynomials satisfy the 3-term recurrence relation

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x). \tag{6.4}$$

It can be verified that $L_n(x) = \left[\frac{-1}{t-1}, \frac{t}{t-1}\right].e^{xt} \equiv \left[\frac{1}{1-t}, \frac{-t}{1-t}\right].e^{xt}$. The general formula for the polynomial sequence $L_n(x)$ is computed using the FTRA (1.5.1) and the method of coefficients as follows:

$$\begin{aligned}
[t^n]\left[\frac{1}{1-t}, \frac{-t}{1-t}\right].e^{xt} &= n!\,[t^n]\,\frac{1}{1-t}e^{-\frac{xt}{1-t}} \\
&= n!\,[t^n]\,\frac{1}{1-t}\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^k}{k!(1-t)^k} \\
&= n!\,[t^n]\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^k}{k!}(1-t)^{-(k+1)} \\
&= n!\,[t^n]\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^k}{k!}\sum_{j=0}^{\infty}\binom{-(k+1)}{j}(-t)^j \\
&= n!\,[t^n]\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^k}{k!}\sum_{j=0}^{\infty}\binom{k+1-1}{j}(-1)^j t^j \\
&= n!\,[t^n]\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^k}{k!}\sum_{j=0}^{\infty}\binom{k}{j}(-1)^j(-1)^j t^j \\
&= \left[t^{n-k}\right]\sum_{k=0}^{n}\frac{n!(-1)^k x^k}{k!}\binom{k}{n-k}t^{n-k} \\
&= \sum_{k=0}^{n}\frac{n!(-1)^k}{k!}\binom{n}{k}x^k \\
&= (L_n(x))_{n\in\mathbf{N}}.
\end{aligned}$$

The coefficient matrix of the Laguerre polynomials $L_n(x)$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -4 & 1 & 0 & 0 & 0 \\ 6 & -18 & 9 & -1 & 0 & 0 \\ 24 & -96 & 72 & -16 & 1 & 0 \\ 120 & -600 & 600 & -200 & 25 & -1 \end{pmatrix}.$$

The production matrix associated to A is given by

$$P_A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -4 & 5 & -1 & 0 \\ 0 & 0 & -9 & 7 & -1 \\ 0 & 0 & 0 & -16 & 9 \end{pmatrix}.$$

Remark: The production matrix P_A is tridiagonal which indicates that A^{-1} forms a family of orthogonal polynomial sequences.

We verify that the polynomial sequence $L_n(x)$ forms an orthogonal polynomial sequence satisfying the 3-term recurrence relation (6.4). We proceed by calculating the inverse of the matrix A corresponding to the coefficient matrix $L_n(x)$ and the production matrix of A^{-1} to determine its recurrence coefficients.

The coefficient matrix A^{-1} is given by

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -4 & 1 & 0 & 0 & 0 \\ 6 & -18 & 9 & -1 & 0 & 0 \\ 24 & -96 & 72 & -16 & 1 & 0 \\ 120 & -600 & 600 & -200 & 25 & -1 \end{pmatrix}.$$

The production matrix of A^{-1} is given by

$$P_{A^{-1}} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -4 & 5 & -1 & 0 \\ 0 & 0 & -9 & 7 & -1 \\ 0 & 0 & 0 & -16 & 9 \end{pmatrix}.$$

Remark: The tri-diagonal nature of the production matrix $P_{A^{-1}}$ of A^{-1} shows that the matrix A is the coefficient matrix of a family of orthogonal polynomial sequences.

Recall from section (1.7) that for a polynomial sequence $(p_n(x))_{n\in\mathbb{N}}$ satisfies the 3-term recurrence relation

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x) \text{ for } n \geq 1.$$

From the production matrix $P_{A^{-1}}$ we get $\alpha_n = 2n + 1$ and $\beta_n = -n^2$ with the initial conditions $L_0(x) = 1$ and $L_1(x) = 1 - x$. Thus,

$$L_{n+1}(x) = (2n + 1 - x)L_n(x) - n^2 L_{n-1}(x)$$

for $n \geq 1$.

6.1.2 Hermite Polynomials

The Hermite differential equation is a second order differential equation given by

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0. \tag{6.5}$$

The solutions to the Hermite equation (6.5) are called the **Hermite polynomials.** There are two forms of the Hermite polynomials. These are the **probabilist Hermite polynomials** and the **physicist Hermite polynomials** denoted He_n annd H_n respectively. The relationship between the two types of Hermite polynomials is such that

$$He_n(x) = 2^{-\frac{n}{2}} H_n(\sqrt{2}x).$$

6.1.2.1 The Probabilist Hermite polynomials He_n

The generating function of the probabilist Hermite polynomials is given by

$$e^{xt - \frac{t^2}{2}} = \sum_{n=0}^{\infty} He_n(x)\frac{t^n}{n!}.$$

The probabilists Hermite polynomial $He_n(x)$ can be defined as

$$He_n(x) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k x^{n-2k}}{k!(n-2k)!2^k} \equiv \sum_{k=0}^{n} \frac{n!}{(-2)^{\frac{n-k}{2}} k!(\frac{n-k}{2})!} \frac{1 + (-1)^{n-k}}{2} x^k.$$

The recursion relation of the Hermite polynnomials $He_n(x)$ is given by

$$He_{n+1}(x) = xHe_n(x) - nHe_{n-1}(x). \tag{6.6}$$

The coefficients of the Hermite polynomials He_n can be represented by the Riordan array $\left[e^{\frac{-t^2}{2}}, t\right]$. such that

$$\left[e^{\frac{-t^2}{2}}, t\right].e^{xt} = (He_n(x))_{n\in\mathbb{N}}.$$

The general formula for the polynomial sequence $He_n(x)$ is computed using the FTRA (1.5.1) and the method of coefficients as follows.

$$\begin{aligned}
[t^n]\left[e^{\frac{-t^2}{2}}, t\right].e^{xt} &= n![t^n]e^{-\frac{t^2}{2}+xt} \\
&= n![t^n]\sum_{k=0}^{\infty}\frac{(-\frac{t^2}{2}+xt)^k}{k!} \\
&= \frac{n!}{k!}[t^n]\sum_{k=0}^{\infty}t^k\left(\frac{-t}{2}+x\right)^k \\
&= \frac{n!}{k!}[t^n]\sum_{k=0}^{\infty}t^k\left(t\left(-\frac{1}{2}+\frac{x}{t}\right)\right)^k \\
&= \frac{n!}{k!}[t^n]\sum_{k=0}^{\infty}t^{2k}\left(-\frac{1}{2}+\frac{x}{t}\right)^k \\
&= \frac{n!}{k!}[t^n]\sum_{k=0}^{\infty}t^{2k}\left(-\frac{1}{2}\left(1-\frac{2x}{t}\right)\right)^k \\
&= \frac{n!}{2^k k!}[t^n]\sum_{k=0}^{\infty}t^{2k}(-1)^k\left(1-\frac{2x}{t}\right)^k \\
&= \frac{n!}{2^k k!}[t^n]\sum_{k}t^{2k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}\left(\frac{-2x}{t}\right)^j \\
&= \frac{n!}{2^k k!}[t^n]\sum_{k}t^{2k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{2x}{t}\right)^j \\
&= \frac{n!}{2^k k!}[t^n]\sum_{k}(-1)^k t^{2k-j}\sum_{j}\binom{k}{j}(-1)^j(2x)^j \\
&= \frac{n!}{2^k k!}\sum_{k=0}(-1)^k\binom{k}{2k-n}(-1)^{2k-n}(2x)^{2k-n}.
\end{aligned}$$

The coefficient matrix of the Hermite polynomials $He_n(x)$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & -6 & 0 & 1 & 0 & 0 \\ 0 & 15 & 0 & -10 & 0 & 1 & 0 \\ -15 & 0 & 45 & 0 & -15 & 0 & 1 \end{pmatrix}.$$

The production matrix associated to A is given by

$$P_A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & -5 & 0 \end{pmatrix}.$$

Remark: The production matrix P_A is tridiagonal which indicates that A^{-1} forms a family of orthogonal polynomial sequences.

We verify that the polynomial sequence $He_n(x)$ forms an orthogonal polynomial sequence satisfying the 3-term recurrence relation (6.6). We proceed by calculating the inverse of the matrix A corresponding to the coefficient matrix $He_n(x)$ and the production matrix of A^{-1} to determine its recurrence coefficients.

The coefficient matrix A^{-1} is given by

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 6 & 0 & 1 & 0 & 0 \\ 0 & 15 & 0 & 10 & 0 & 1 & 0 \\ 15 & 0 & 45 & 0 & 15 & 0 & 1 \end{pmatrix}.$$

The production matrix of A^{-1} is given by

$$P_{A^{-1}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{pmatrix}.$$

Remark: The tri-diagonal nature of the production matrix $P_{A^{-1}}$ of A^{-1} shows that the matrix A is the coefficient matrix of a family of orthogonal polynomial sequences defined by the Hermite polynomials $He_n(x)$. Recall from section (1.7) that for an orthogonal polynomial sequence $(p_n(x))_{n\in\mathbb{N}}$ satisfies the 3-term recurrence relation

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x),\ n \geq 1\ .$$

From the production matrix $P_{A^{-1}}$ we get $\alpha_n = 0$ and $\beta_n = n$. Thus,

$$He_{n+1}(x) = xHe_n(x) - nHe_{n-1}(x)$$

for $n \geq 1$ with the initial conditions $He_0 = 1$ and $He_1 = x$.

6.1.2.2 The Physicist Hermite polynomials $H_n(x)$

The **physicists Hermite polynomials** $H_n(x)$ may be defined as

$$H_n(x) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}.$$

The generating function for $H_n(x)$ is given by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

The recursion relation of the Hermite polynomials $H_n(x)$ is given by

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{6.7}$$

The coefficients of the Hermite polynomials $H_n(x)$ can be represented by the Riordan array $\left[e^{-t^2}, 2t\right]$ such that

$$\left[e^{-t^2}, 2t\right].e^{xt} = (H_n(x))_{n\in\mathbb{N}}\ .$$

The general formula for the polynomial sequence $H_n(x)$ is computed using the FTRA (1.5.1) and the method of coefficients as follows:

$$
\begin{aligned}
[t^n]\left[e^{-t^2}, 2t\right].e^{xt} &= n![t^n]e^{-t^2+2xt} \\
&= n![t^n]\sum_{k=0}^{\infty}\frac{(-t^2+2xt)^k}{k!} \\
&= \frac{n!}{k!}[t^n]\sum_{k=0}^{\infty} t^k\left(-t+2x\right)^k \\
&= \frac{n!}{k!}[t^n]\sum_{k=0}^{\infty} t^k\left(t\left(-1+\frac{2x}{t}\right)\right)^k \\
&= \frac{n!}{k!}[t^n]\sum_{k=0}^{\infty} t^{2k}\left(-1+\frac{2x}{t}\right)^k \\
&= \frac{n!}{k!}[t^n]\sum_{k=0}^{\infty} t^{2k}\left(-1\left(1-\frac{2x}{t}\right)\right)^k \\
&= \frac{n!}{k!}[t^n]\sum_{k=0}^{\infty} t^{2k}(-1)^k\left(1-\frac{2x}{t}\right)^k \\
&= \frac{n!}{k!}[t^n]\sum_{k} t^{2k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}\left(\frac{-2x}{t}\right)^j \\
&= \frac{n!}{k!}[t^n]\sum_{k} t^{2k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{2x}{t}\right)^j \\
&= \frac{n!}{k!}[t^n]\sum_{k} t^{2k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{2x}{t}\right)^j \\
&= \frac{n!}{k!}[t^n]\sum_{k}(-1)^k t^{2k-j}\sum_{j}\binom{k}{j}(-1)^j(2x)^j \\
&= \frac{n!}{k!}\sum_{k=0}(-1)^k\binom{k}{2k-n}(-1)^{2k-n}(2x)^{2k-n}..
\end{aligned}
$$

The coefficient matrix of the Hermite polynomials $H_n(x)$ is given by

$$
A=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & 0 & 0\\
-2 & 0 & 4 & 0 & 0 & 0 & 0\\
0 & -12 & 0 & 8 & 0 & 0 & 0\\
12 & 0 & -48 & 0 & 16 & 0 & 0\\
0 & 120 & 0 & -160 & 0 & 32 & 0\\
-120 & 0 & 720 & 0 & -480 & 0 & 64
\end{pmatrix}.
$$

The production matrix associated to A is given by

$$P_A = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & -4 & 0 & 2 \\ 0 & 0 & 0 & 0 & -5 & 0 \end{pmatrix}.$$

Remark: The production matrix P_A is tridiagonal which indicates that A^{-1} forms a family of orthogonal polynomial sequences.

We verify that the polynomial sequence $H_n(x)$ forms an orthogonal polynomial sequence satisfying the 3-term recurrence relation (6.7). We proceed by calculating the inverse of the matrix A corresponding to the coefficient matrix $H_n(x)$ and the production matrix of A^{-1} to determine its recurrence coefficients.

The coefficient matrix A^{-1} is given by

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{8} & 0 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{3}{4} & 0 & \frac{1}{16} & 0 & 0 \\ 0 & \frac{15}{8} & 0 & \frac{5}{8} & 0 & \frac{1}{32} & 0 \\ \frac{15}{8} & 0 & \frac{45}{16} & 0 & \frac{15}{32} & 0 & \frac{1}{64} \end{pmatrix}.$$

The production matrix of A^{-1} is given by

$$P_{A^{-1}} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 3 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 4 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 5 & 0 \end{pmatrix}.$$

Remark: The tri-diagonal nature of the production matrix $P_{A^{-1}}$ of A^{-1} shows that the matrix A is the coefficient matrix of a family of orthogonal polynomial sequences defined by the Hermite polynomials $H_n(x)$.

Recall from section (1.7) that for an orthogonal polynomial sequence $(p_n(x))_{n\in\mathbb{N}}$ satisfies the 3-term recurrence relation

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x).$$

From the production matrix $P_{A^{-1}}$ we get $\alpha_n = 0$ and $\beta_n = 2n$ (by applying a scaling factor of 2 to $P_{A^{-1}}$ gives

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

as defined in (6.7).

6.1.3 The Reverse Bessel Polynomials

The reverse Bessel polynomial satisfies the differential equation given by

$$x\theta_n''(x) - 2(x+n)\theta_n'(x) + 2n\theta_n(x) = 0. \tag{6.8}$$

The explicit formula of the reverse Bessel polynomial is given by

$$\theta_n(x) = \sum_{k=0}^{n} \frac{(2n-k)!}{2^{n-k}k!(n-k)!}x^k. \tag{6.9}$$

where $k = 0, 1, ..., n$.
The 3− term recurrence relation of the reverse Bessel polynomial is given by

$$\theta_n(x) = (2n-1)\theta_{n-1}(x) + x^2\theta_{n-2}(x).$$

The proper exponential Riordan array of the reverse Bessel polynomial is $R_{BS} = \left[\frac{1}{\sqrt{1-2t}}, 1-\sqrt{1-2t}\right]$ represents the solution of (6.8), in the sense that

$$R_{BS} \cdot \{1, \frac{x}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots\}^T = \{\theta_0(x), \theta_1(x), \theta_2(x), \ldots\}^T.$$

The coefficient matrix of the reverse Bessel polynomial is given by

$$R_{BS} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 0 \\ 15 & 15 & 6 & 1 & 0 & 0 \\ 105 & 105 & 45 & 10 & 1 & 0 \\ 945 & 945 & 420 & 105 & 15 & 1 \end{pmatrix}.$$

The production matrix of R_{BS} is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 6 & 6 & 3 & 1 & 0 \\ 24 & 24 & 12 & 4 & 1 \\ 120 & 120 & 60 & 20 & 5 \end{pmatrix}.$$

The general formula for the entries of the coefficient matrix of the Reverse Bessel polynomials are calculated using its Riordan array representation as follows:

$$\begin{aligned}
\left[\frac{1}{\sqrt{1-2x}}, 1-\sqrt{1-2x}\right] &= \frac{n!}{k!}[x^n]\frac{1}{\sqrt{1-2x}}\left(1-\sqrt{1-2x}\right)^k \\
&= \frac{n!}{k!}[x^n]\frac{1}{(1-2x)^{1/2}}\sum\binom{k}{j}(-1)^j(1-2x)^{j/2} \\
&= \frac{n!}{k!}[x^n]\sum_{j=0}^{k}\binom{k}{j}(-1)^j(1-2x)^{\frac{j-1}{2}} \\
&= \frac{n!}{k!}[x^n]\sum_{j=0}^{k}\binom{k}{j}(-1)^j\sum_i\binom{\frac{j-1}{2}}{i}(-2)^i x^i \\
&= \frac{n!}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\binom{\frac{j-1}{2}}{n}(-2)^n \\
&= \frac{n!}{k!}(-2)^n\sum_{j=0}^{k}\binom{k}{j}\binom{\frac{j-1}{2}}{n}(-1)^j \\
&= \frac{n!}{k!}(-2)^n\sum_{j=0}^{k}\binom{k}{j}(-1)^j\binom{\frac{j-1}{2}}{n}.
\end{aligned}$$

The *Mathematica* code below computes the coefficient matrix of the reverse Bessel polynomials using the general formula.

```
Table[Table[n!/k! (-2)^n Sum[Binomial[k, j] Binomial[(j-1)/2, n] (-1)^j, {j, 0, k}], {k, 0, 10}],
{n, 0, 10}]
```

6.1.4 Chebyshev Polynomials

The Chebyshev differential equation given by

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \lambda^2 y = 0. \tag{6.10}$$

In order to determine that the Chebyshev equation satisfies the Sturm-Liouville equation (6.1) it is divided by $\sqrt{1-x^2}$ to get the equation

$$(\sqrt{1-x^2})y'' - \frac{x}{\sqrt{1-x^2}}y' + \frac{\lambda^2}{\sqrt{1-x^2}}y = 0 \tag{6.11}$$

such that $(\sqrt{1-x^2})y'' - \frac{x}{\sqrt{1-x^2}}y'$ corresponds to $\frac{d}{dx}\left[(\sqrt{1-x^2})\frac{dy}{dx}\right]$ in the context of the Sturm-Liouville form (6.1) with

$$p(x) = \sqrt{1-x^2},\ q(x) = 0,\ w(x) = \frac{1}{\sqrt{1-x^2}}.$$

The solutions to the Chebyshev equation are known as the **Chebyshev polynomials** The Chebyshev polynomials are of two types referred to as the **Chebyshev polynomials of the first kind** and the **Chebyshev polynomials of the second kind**.

6.1.4.1 The Chebyshev polynomials of the second kind

The generating function of the Chebyshev polynomial of the second kind[122] is

$$u(t,x) = \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n. \tag{6.12}$$

The formula for the general term of the chebyschev polynomial of the second kind is given by

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}(-1)^k 2x^{n-2k}.$$

The ordinary Riordan array $R_c = \left(\frac{1}{1+t^2}, \frac{2t}{1+t^2}\right)$ represents the solution of the differential system in the sense that

$$R_c \cdot \{1, x, x^2, \ldots\}^T = \{U_0(x), U_1(x), U_2(x), \ldots\}^T.$$

6.1.4.2 The Scaled Chebyshev Polynomials

The family of scaled Chebyshev polynomials

$$y_n(x) = U_n(x/2) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}(-1)^k x^{n-2k}$$

satisfies the system of differential equations

$$(4-x^2)y_n'' - 3xy_n' + n(n+2)y_n = 0.$$

The Riordan array

$$R = \left(\frac{1}{1+t^2}, \frac{t}{1+t^2} \right)$$

represents the solution, in the sense that

$$R \cdot \{1, x, x^2, \ldots\}^T = \{y_0(x), y_1(x), y_2(x), \ldots\}^T.$$

The general formula for the polynomial sequence $y_n(x)$ is computed using the FTRA (1.5.1) and the method of coefficients as follows:

$$\begin{aligned}
[t^n] \left(\frac{1}{1+t^2}, \frac{t}{1+t^2} \right) \cdot \frac{1}{1-xt} &= [t^n] \frac{\frac{1}{1+t^2}}{1 - x\left(\frac{t}{1+t^2}\right)} \\
&= [t^n] \frac{1}{1+t^2} \left(\frac{1}{1 - x\left(\frac{t}{1+t^2}\right)} \right) \\
&= [t^n] \frac{1}{1+t^2} \sum_{k=0}^{\infty} \left(\frac{xt}{1+t^2} \right)^k \\
&= [t^n] \frac{1}{1+t^2} \sum_{k=0}^{\infty} \frac{t^k}{(1+t^2)^k} x^k \\
&= [t^n] \sum_{k=0}^{\infty} \frac{t^k}{(1+t^2)^{k+1}} x^k \\
&= [t^{n-k}] \sum_{j=0}^{\infty} \binom{-(k+1)}{j} t^{2j} x^k \\
&= [t^{n-k}] \sum_{j=0}^{\infty} \binom{k+1+j-1}{j} (-1)^j t^{2j} x^k \\
&= [t^{n-k}] \sum_{j=0}^{\infty} \binom{k+j}{j} (-1)^j t^{2j} x^k \\
&= \sum_{k=0}^{n} \binom{\frac{n+k}{2}}{k} (-1)^{\frac{n-k}{2}} \frac{(1+(-1)^{n-k})}{2} x^k \\
&= (y_n(x))_{n\in\mathbb{N}}.
\end{aligned}$$

Remark: From the above results we have $2j = n-k \implies j = \frac{n-k}{2}$ so that

$$\binom{k + \frac{n-k}{2}}{\frac{n-k}{2}} (-1)^{\frac{n-k}{2}} \frac{(1+(-1)^{n-k})}{2}$$

and also

$$\frac{n+k}{2} - \frac{n-k}{2} = k.$$

6.1.5 Morgan-Voyce Polynomials

The Morgan-Voyce polynomials are can be explicitly defined in two types denoted by $B_n(x)$ and $b_n(x)$.

6.1.5.1 The Morgan-Voyce Polynomials $B_n(x)$

The Morgan-Voyce polynomial $B_n(x)$ is given explicitly by the formula

$$B_n(x) = \sum_{k=0}^{n} \binom{n+k+1}{n-k} x^k \tag{6.13}$$

and it satisfies the ordinary differential equation

$$x(x+4)y'' + 3(x+2)y' - n(n+2)y = 0. \tag{6.14}$$

The Morgan-Voyce polynomials $B_n(x)$ can also be defined by the 3-term recurrence relation

$$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x) \tag{6.15}$$

with $B_0(x) = 1$ and $B_1(x) = 2 + x$.

The proper ordinary Riordan array $R_{MB} = \left(\frac{1}{(1-t)^2}, \frac{t}{(1-t)^2}\right)$ represents the solution of (6.14), in the sense that

$$R_{MB} \cdot \{1, x, x^2, \ldots\}^T = \{B_0(x), B_1(x), B_2(x), \ldots\}^T.$$

The general formula for the polynomial sequence $B_n(x)$ is computed using the

FTRA (1.5.1) and the method of coefficients as follows:

$$
\begin{aligned}
[t^n]\left(\frac{1}{(1-t)^2}, \frac{t}{(1-t)^2}\right)\cdot\frac{1}{1-xt} &= [t^n]\frac{\frac{1}{(1-t)^2}}{1-x\left(\frac{t}{(1-t)^2}\right)} \\
&= [t^n]\frac{1}{(1-t)^2}\left(\frac{1}{1-x\left(\frac{t}{(1-t)^2}\right)}\right) \\
&= [t^n]\frac{1}{(1-t)^2}\sum_{k=0}^{\infty}\left(\frac{xt}{(1-t)^2}\right)^k \\
&= [t^n]\frac{1}{(1-t)^2}\sum_{k=0}^{\infty}x^k\cdot\frac{t^k}{(1-t)^{2k}} \\
&= [t^n]\sum_{k=0}^{\infty}x^k\cdot t^k(1-t)^{-(2k+2)} \\
&= [t^{n-k}]\sum_{j=0}^{\infty}\binom{2k+2+j-1}{j}(-1)^j(-t)^j x^k \\
&= [t^{n-k}]\sum_{k=0}^{n}\binom{2k+n-k+1}{n-k}t^{n-k}x^k \\
&= \sum_{k=0}^{n}\binom{n+k+1}{n-k}x^k \\
&= (B_n(x))_{n\in\mathbb{N}}.
\end{aligned}
$$

The coefficient matrix of the Morgan-Voyce polynomials $B_n(x)$ is given by

$$
A=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 1 & 0 & 0 & 0 & 0 & 0\\
3 & 4 & 1 & 0 & 0 & 0 & 0\\
4 & 10 & 6 & 1 & 0 & 0 & 0\\
5 & 20 & 21 & 8 & 1 & 0 & 0\\
6 & 35 & 56 & 36 & 10 & 1 & 0\\
7 & 56 & 126 & 120 & 55 & 12 & 1
\end{pmatrix}.
$$

The production matrix associated to A is given by

$$
P_A=\begin{pmatrix}
2 & 1 & 0 & 0 & 0\\
-1 & 2 & 1 & 0 & 0\\
2 & -1 & 2 & 1 & 0\\
-5 & 2 & -1 & 2 & 1\\
14 & -5 & 2 & -1 & 2
\end{pmatrix}.
$$

Remark: The production matrix P_A corresponds to the structural pattern of the production matrices for proper ordinary Riordan arrays defined in section (1.6.1).

We verify that the polynomial sequence $B_n(x)$ forms an orthogonal polynomial sequence satisfying the 3-term recurrence relation (6.15). We proceed by calculating the inverse of the matrix A corresponding to the coefficient matrix $B_n(x)$ and the production matrix of A^{-1} to determine its recurrence coefficients.

The coefficient matrix A^{-1} is given by

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & -4 & 1 & 0 & 0 & 0 & 0 \\ -14 & 14 & -6 & 1 & 0 & 0 & 0 \\ 42 & -48 & 27 & -8 & 1 & 0 & 0 \\ -132 & 165 & -110 & 44 & -10 & 1 & 0 \\ 429 & -572 & 429 & -208 & 65 & -12 & 1 \end{pmatrix}.$$

The production matrix of A^{-1} is given by

$$P_{A^{-1}} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Remark: The tri-diagonal nature of the production matrix $P_{A^{-1}}$ of A^{-1} shows that the matrix A is the coefficient matrix of a family of orthogonal polynomial sequences defined by the Morgan-Voyce polynomials $B_n(x)$.

Recall from section (1.7) that for an orthogonal polynomial sequence $(p_n(x))_{n\in\mathbb{N}}$ satisfies the 3-term recurrence relation

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x).$$

From the production matrix $P_{A^{-1}}$ we get $\alpha_n = -2$ and $\beta_n = 1$. Thus,

$$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x) \text{ with } B_0(x) = 1 \text{ and } B_1(x) = 2 + x$$

as defined in (6.15).

6.1.5.2 The Morgan-Voyce Polynomials $b_n(x)$

The Morgan-Voyce polynomials $b_n(x)$ is given explicitly by

$$b_n(x) = \sum_{k=0}^{n} \binom{n+k}{n-k} x^k \tag{6.16}$$

and it satisfies the equation

$$x(x+4)y'' + 2(x+1)y' - n(n+1)y = 0. \tag{6.17}$$

The Morgan-Voyce polynomials $b_n(x)$ can also be defined by the 3-term recurrence relation

$$b_n(x) = (x+2)b_{n-1}(x) - b_{n-2}(x) \tag{6.18}$$

with $b_0(x) = 1$ and $b_1(x) = 1 + x$. The proper ordinary Riordan array $R_{Mb} = \left(\frac{1}{1-t}, \frac{t}{(1-t)^2}\right)$ represents the solution of (6.17), in the sense that

$$R_{Mb} \cdot \{1, x, x^2, \ldots\}^T = \{b_0(x), b_1(x), b_2(x), \ldots\}^T.$$

The general formula for the polynomial sequence $b_n(x)$ is computed using the FTRA (1.5.1) and the method of coefficients as follows:

$$\begin{aligned}
[t^n]\left(\frac{1}{1-t}, \frac{t}{(1-t)^2}\right)\cdot\frac{1}{1-xt} &= [t^n]\frac{\frac{1}{1-t}}{1-x\left(\frac{t}{(1-t)^2}\right)} \\
&= [t^n]\frac{1}{1-t}\left(\frac{1}{1-x\left(\frac{t}{(1-t)^2}\right)}\right) \\
&= [t^n]\frac{1}{1-t}\sum_{k=0}^{\infty}\left(\frac{xt}{(1-t)^2}\right)^k \\
&= [t^n]\frac{1}{1-t}\sum_{k=0}^{\infty}x^k\cdot\frac{t^k}{(1-t)^{2k}} \\
&= [t^n]\sum_{k=0}^{\infty}x^k\cdot t^k(1-t)^{-(2k+1)} \\
&= [t^{n-k}]\sum_{j=0}^{\infty}\binom{2k+1+j-1}{j}(-1)^j(-t)^j x^k \\
&= [t^{n-k}]\sum_{k=0}^{n}\binom{2k+n-k}{n-k}t^{n-k}x^k \\
&= \sum_{k=0}^{n}\binom{n+k}{n-k}x^k \\
&= (b_n(x))_{n\in\mathbb{N}}.
\end{aligned}$$

The coefficient matrix of the Morgan-Voyce polynomials $b_n(x)$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 5 & 1 & 0 & 0 & 0 \\ 1 & 10 & 15 & 7 & 1 & 0 & 0 \\ 1 & 15 & 35 & 28 & 9 & 1 & 0 \\ 1 & 21 & 70 & 84 & 45 & 11 & 1 \end{pmatrix}.$$

The production matrix associated to A is given by

$$P_A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 2 & -1 & 2 & 1 \\ 0 & -5 & 2 & -1 & 2 \end{pmatrix}.$$

Remark: The production matrix P_A corresponds to the structural pattern of the production matrices for proper ordinary Riordan arrays defined in section (1.6.1).

We verify that the polynomial sequence $b_n(x)$ forms an orthogonal polynomial sequence satisfying the 3-term recurrence relation (6.18). We proceed by calculating the inverse of the matrix A corresponding to the coefficient matrix $b_n(x)$ and the production matrix of A^{-1} to determine its recurrence coefficients.

The coefficient matrix A^{-1} is given by

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 & 0 \\ -5 & 9 & -5 & 1 & 0 & 0 & 0 \\ 14 & -28 & 20 & -7 & 1 & 0 & 0 \\ -42 & 90 & -75 & 35 & -9 & 1 & 0 \\ 132 & -297 & 275 & -154 & 54 & -11 & 1 \end{pmatrix}.$$

The production matrix of A^{-1} is given by

$$P_{A^{-1}} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Remark: The tri-diagonal nature of the production matrix $P_{A^{-1}}$ of A^{-1} shows that the matrix A is the coefficient matrix of a family of orthogonal polynomial sequences defined by the Morgan-Voyce polynomials $b_n(x)$.

Recall from section (1.7) that for an orthogonal polynomial sequence $(p_n(x))_{n\in\mathbb{N}}$ satisfies the 3-term recurrence relation

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x).$$

From the production matrix $P_{A^{-1}}$ we get $\alpha_n = -2$ and $\beta_n = 1$. Thus,

$$b_n(x) = (x+2)b_{n-1}(x) - b_{n-2}(x) \text{ with } b_0(x) = 1 \text{ and } b_1(x) = 1 + x$$

as defined in (6.18).

6.2 Boubaker polynomials and the solutions of some Differential Equations

The Boubaker polynomials form a monic orthogonal sequence of polynomials which originated from an attempt to determine the discrete form of solution to a non-linear heat transfer problem related to the spray pyrolysis disposal performance [16]. This led to the development of the Boubaker polynomials expansion scheme denoted BPES, which is a resolution protocol with several applications in applied physics, engineering and mathematical problems. Some collection of the applications of BPES have been listed and summarized in the 3 page paper [17].

The main advantage of the BPES is that it requires that the boundary conditions are satisfied irrespective of the main features of the equation. The BPES is based on the Boubaker polynomials. The recursive definition of the Boubaker polynomials is given by

$$B_n(x) = \begin{cases} 1 & \text{if } n = 0 \\ x & \text{if } n = 1 \\ x^2{+}2 & \text{if } n = 2 \\ xB_{n-1}(x)\text{-}B_{n-2}(x) & \text{otherwise, } n \in \mathbb{Z}_{\geq 2} \end{cases}$$

The ordinary generating function of the Boubaker polynomials is given by

$$B(x,t) = \frac{1+3t^2}{1+t(t-x)}.$$

The characteristic differential equation of the Boubaker polymials is given by

$$A_n y'' + B_n y' - C_n y = 0$$

where

$$\begin{aligned} A_n &= (x^2-1)(3nx^2+n-2) \\ B_n &= 3x(nx^2+3n-2) \\ C_n &= -n(3n^2x^2+n^2-6n+8). \end{aligned}$$

The explicit closed form formula of the Boubaker polynomials is given by

$$B_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n-4k}{n-k} \binom{n-k}{K} (-1)^k x^{n-2k}.$$

The first 5 Boubaker polynomials are $B_0(x) = 1$
$B_1(x) = x - 0x^0$
$B_2(x) = x^2 - 0x + 2x^0$
$B_3(x) = x^3 - 0x^0 + x + 0$
$B_4(x) = x^4 - 0x^3 + 0x^2 + 0x - 2$
$B_5(x) = x^5 + 0x^4 - x^3 + 0x^2 - 3x + 0x^0$

The relationship between the Boubaker polynomials and the Chebyshev polynomials [92] is given by

$$B_n(2x) = \frac{4x}{n}\frac{d}{dx}T_n(x) - 2T_n(x)$$

$$B_n(2x) = -2T_n(x) + 4xU_{n-1}(x)$$

where T_n denotes the Chebyshev polynomials of the first kind and U_n denotes the Chebyshev polynomials of the second kind.

The Riordan array representation of the coefficients of the Boubaker polynomials is given by

$$B = \left(\frac{1+3x^2}{1+x^2}, \frac{x}{1+x^2}\right).$$

The coefficient matrix of B is given by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & -1 & 0 & 1 & 0 \\ 2 & 0 & -3 & 0 & -2 & 0 & 1 \end{pmatrix}.$$

The production matrix of B is given by

$$P_B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -4 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 4 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}.$$

The inverse of the matrix B is given by

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 \\ -4 & 0 & 3 & 0 & 2 & 0 & 1 \end{pmatrix}.$$

The production matrix of B^{-1} is given by

$$P_{B^{-1}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The tri-diagonal nature of the production matrix $P_{B^{-1}}$ of B^{-1} shows that the matrix B is the coefficient matrix of a family of orthogonal polynomial sequences defined by the Boubaker polynomials $B_n(x)$. Recall from section (1.7) that for an orthogonal polynomial sequence $(p_n(x))_{n\in\mathbb{N}}$ satisfies the 3-term recurrence relation

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x).$$

From the production matrix $P_{B^{-1}}$ we get $\alpha_n = 0$ and $\beta_n = 1$. Thus, we have

$$B_n(x) = xB_{n-1}(x) - B_{n-2}(x)$$

where the initial conditions $B_0 = 1, B_1(x) = x, B_2(x) = x^2 + 2$ are satisfied.

The 4 key properties derived from the Boubaker polynomials pertains to its polynomial non-zero orders being divisible by 4. This forms the basis behind its already existing and ongoing research to analytic solutions of the differential equations that it solves. These properties when considered for $n = 4q$, $q = 1.....N$, $N \in \mathbb{Z}_{>0}$ are listed as:

1. $\sum_{n>0} B_n(x) \mid_{x=0} = -2N \neq 0.$

2. $\sum_{n>0} B_n(\alpha_i) = 0$, α_i a non-zero positive root of B_n.

3. $\sum_{n>0} \frac{dB_n(x)}{dx} = 0 \mid_{x=0}$ and $H_n = B'_{4k}(r_n) = \left(\frac{4r_n\left[2-r_n^2\right] \sum_{q=1}^{n} B_{4q}^2(r_n)}{B_{4(n+1)}(r_n)} + 4r_n^3 \right).$

4. $\sum_{n>0} \frac{dB_n^2(x)}{dx^2} = \frac{8}{3}N(N^2-1) \mid_{x=0}.$

Some of the differential equations having analytic solutions arising from the implementation of the BPES technique are listed below.

- The Lane-Emden equations of the first and second kind respectively governing polytropic and isothermal gas spheres, are given in Boubaker & Gorder[18] as follows

$$y'' + \frac{2}{x}y' + y^n = 0 \text{ with initial conditions } y(0) = 1,\ y'(0) = 0.$$

$$y'' + \frac{2}{x}y' = e^{-y} \text{ with initial conditions } y(0) = 0,\ y'(0) = 0.$$

- The Bloch NMR Flow equation given in Awojoyogbe et al.[4] as follows

$$x\frac{d^2M_y}{dx^2} + \alpha\frac{dM_y}{dx} + \beta x M_y^n = 0 \mid_{\alpha=1,\beta=2;n=2,3,4,5}.$$

  satisfying the initial conditions

$$M_y(0) = 1; \frac{dM_y(0)}{dx} = 0$$

  where β is a unique constant associated to the NMR system under investigation.

- A differential equation from a nonlinear circuit investigated in Vazquez et al.[113] is given by

$$\frac{dq(t)}{dt} + \frac{\alpha}{R}q^3(t) = I, \qquad q(0) = 0.$$

Other examples of differential equations solved using the BPES technique are found in [59, 2, 112].

Chapter 7

Elliptic Functions and the Travelling Wave Solutions to the KdV equation

7.1 Introduction

The presence of solitary water waves moving for a considerable distance down a narrow channel was first observed and reported by John Scott Russell, a Scottish naval engineer back in 1834. This was followed up by Korteweg and de Vries(1895) mathematical formulation of a third order nonlinear PDE referred to as the KdV equation [60]. The role of this KdV equation which admits traveling wave solutions was to determine an approximate theory to model the evolution of the propagation of nonlinear shallow water waves. There are many different variants of the KdV equation. The standard form of the KdV equation is given by

$$u_t - 6uu_x + u_{xxx} = 0. \tag{7.1}$$

Sometimes it is $+6$ used in the above equation. In the KdV equation, the variable t denotes the time, the variable x denotes the space coordinate along the canal and the function to be determined denoted $u = u(x,t)$ represents the elevation of the fluid above the bottom of the canal [3].

In 1965 Kruskal and Zabusky showed that the KdV equation admits analytic solutions known as "Solitons" [123]. Solitons can be described as propagating pulses or solitary waves which are stable by maintaining their shape and do not disperse with time. The KdV equation can also have more than one soliton solution which can move towards each other, interact and then emerge at the same speed with no change in shape but only a time lag [93].

There are several methods available for solving different forms of non-linear wave equations in mathematical physics [52]. The method applied in determining the explicit exact solutions to the KdV equation provides a good entry point on how to approach the solutions of other such non-linear wave equations. In particular, using appropriate elliptic functions to determine the explicit solutions for the KdV equation is important as it leads to periodic wave solutions and not only solitary wave solutions.

7.2 Elliptic functions as solutions to the KdV equation

In this section we shall consider the two types of elliptic functions that solve the KdV equation. These are the cnoidal travel wave solution determined by elliptic cn^2 and the elliptic Weierstrass $\wp$. Each of these solutions characterized by their doubly periodic nature are elaborated below.

7.2.1 Elliptic $\mathbf{cn}^2$ as a solution to the KdV equation

In this subsection we verify the solution to the KdV equation (7.1) with the aid of the symbolic computational algebra system *Mathematica*, by using the cnoidal traveling wave solution of the form $u(\xi) = a + bcn^2(\xi, m)$ proposed but unsolved in [36].

Consider the nonlinear PDE of the form

$$H(u, u_t, u_x, u_{xx}, ..) = 0 \tag{7.2}$$

for which equation (7.1) satisfies. H is expressible as a polynomial in u where $u = u(x,t)$ is the unknown function such that u is the physical field and x, t represents its independent variables. The simplest mathematical wave function is of the form $u(x,t) = f(x - ct)$ which can easily be shown to be a solution to the PDE $u_t + cu_x = 0$. In order to obtain an ODE we transform using the simplest traveling wave equation

$$u(x,t) = u(\xi),\ \xi = x - ct \tag{7.3}$$

$u(x,t) = u(\xi), \xi = k(x - ct)$ where c is the wave speed moving in either the left or right direction, k is the wave number.

Using (7.3) the PDE (7.2) can be transformed to a nonlinear ordinary differential equation given by

$$F(u, u', u'', u''', ...) \equiv F\left(u, \frac{du}{d\xi}, \frac{d^2u}{d^2\xi}, \frac{d^3u}{d^3\xi}, \cdots\right)$$

where the prime is such that

$$' := \frac{d}{d\xi}$$

and F is expressible as a polynomial in terms of $u(\xi)$. As a result we have that substituting $u(\xi)$ into (7.1) we get the ODE

$$-c\frac{du}{d\xi} - 6u\frac{du}{d\xi} + \frac{d^3u}{d\xi^3} = 0. \tag{7.4}$$

Integrating equation (7.4), we get the second order differential equation

$$-cu - 3u^2 + \frac{d^2u}{d\xi^2} = k_1 \tag{7.5}$$

where k_1 is the constant of integration. Given that k_1 vanishes if $x \to \infty$ and $u \to 0$, we have

$$-cu - 3u^2 + \frac{d^2u}{d\xi^2} = 0. \tag{7.6}$$

The steps to determine the exact solution of (7.1) are outlined below.

- Using $u(\xi) = a + b\text{cn}^2(\xi, m)$ we have

$$\frac{d^2u}{d\xi^2} = b\left(-2\text{cn}(\xi|m)^2\text{dn}(\xi|m)^2 + 2m\text{cn}(\xi|m)^2\text{sn}(\xi|m)^2 + 2\text{dn}(\xi|m)^2\text{sn}(\xi|m)^2\right). \tag{7.7}$$

- Substituting $u(\xi) = a + b\text{cn}^2(\xi, m)$ and (7.7) into (7.6) we get

$$-c\left(a+b\text{cn}(\xi|m)^2\right)-3\left(a+b\text{cn}(\xi|m)^2\right)^2+$$
$$b\left(-2\text{cn}(\xi|m)^2\text{dn}(\xi|m)^2+2m\text{cn}(\xi|m)^2\text{sn}(\xi|m)^2+2\text{dn}(\xi|m)^2\text{sn}(\xi|m)^2\right). \tag{7.8}$$

- Simplifying and expanding equation (7.8) gives

$$-3a^2-6ab\text{cn}(\xi|m)^2-ac-3b^2\text{cn}(\xi|m)^4-bc\text{cn}(\xi|m)^2-$$
$$4b\text{cn}(\xi|m)^2\text{dn}(\xi|m)^2+2b\text{cn}(\xi|m)^2+2b\text{dn}(\xi|m)^2\text{sn}(\xi|m)^2. \tag{7.9}$$

- Recall that

$$\text{cn}^2\xi=1-\text{sn}^2\xi \;\text{ and }\; \text{dn}^2\xi=1-m^2\text{sn}^2\xi \tag{7.10}$$

where m is the modulus with $0\le m\le 1$.

- Substituting (7.10) for sn^2 and dn^2 in terms of cn^2 in (7.9) results to

$$-3a^2-6ab\text{cn}(\xi|m)^2-ac-3b^2\text{cn}(\xi|m)^4-bc\text{cn}(\xi|m)^2-$$
$$4b\left(1-m^2\left(1-\text{cn}(\xi|m)^2\right)\right)\text{cn}(\xi|m)^2+$$
$$2b\left(1-\text{cn}(\xi|m)^2\right)\left(1-m^2\left(1-\text{cn}(\xi|m)^2\right)\right)+2b\text{cn}(\xi|m)^2. \tag{7.11}$$

- Expanding (7.11) gives

$$-3a^2-6ab\text{cn}(\xi|m)^2-ac-3b^2\text{cn}(\xi|m)^4-bc\text{cn}(\xi|m)^2-6bm^2\text{cn}(\xi|m)^4$$
$$+8bm^2\text{cn}(\xi|m)^2-4b\text{cn}(\xi|m)^2-2bm^2+2b. \tag{7.12}$$

- Extracting the coefficients of (7.12) from cn^i where $i=0,1,2,3,4$ we get

$$\{-3a^2-ac-2bm^2+2b,0,-6ab-bc+8bm^2-4b,0,-3b^2-6bm^2\}. \tag{7.13}$$

- Setting the non-zero elements in (7.13) to zero and solving the system of equations for a,b,c in terms of m, the following set of solutions are obtained :

1. **Solution 1-** If $-\frac{1}{\sqrt{2}} < m < \frac{1}{\sqrt{2}}$ then $a = -\frac{1}{6}\sqrt{16m^4 - 16m^2 + 16} - \frac{1}{6}\sqrt{64m^4 - 64m^2 + 16},\ b = -2m^2,\ c = \sqrt{16m^4 - 16m^2 + 16}$

2. **Solution 2-** If $m > \frac{1}{\sqrt{2}} \vee m < -\frac{1}{\sqrt{2}}$ then $a = \frac{1}{6}\sqrt{64m^4 - 64m^2 + 16} - \frac{1}{6}\sqrt{16m^4 - 16m^2 + 16},\ b = -2m^2,\ c = \sqrt{16m^4 - 16m^2 + 16}$

3. **Solution 3-** If $m > \frac{1}{\sqrt{2}} \vee m < -\frac{1}{\sqrt{2}}$ then $a = \frac{1}{6}\sqrt{16m^4 - 16m^2 + 16} + \frac{1}{6}\sqrt{64m^4 - 64m^2 + 16},\ b = -2m^2,\ c = -\sqrt{16m^4 - 16m^2 + 16}$.

4. Substituting the values of a, b, c obtained in the step above into $u(\xi) = a + b\mathrm{cn}^2(\xi, m)$ where $\xi = x - ct$, we get the following solutions outlined below for (7.1) in the form $u(x,t)$ as required.

   (a) For solution 1 we get $u(x,t) = -2m^2\mathrm{cn}\left(\sqrt{16m^4 - 16m^2 + 16}t - x|\, m\right)^2 - \frac{1}{6}\sqrt{16m^4 - 16m^2 + 16} - \frac{1}{6}\sqrt{64m^4 - 64m^2 + 16}$.

   (b) For solution 2 we get $u(x,t) = -2m^2\mathrm{cn}\left(\sqrt{16m^4 - 16m^2 + 16}t - x|\, m\right)^2 - \frac{1}{6}\sqrt{16m^4 - 16m^2 + 16} + \frac{1}{6}\sqrt{64m^4 - 64m^2 + 16}$.

   (c) For solution 3 we get $u(x,t) = -2m^2\mathrm{cn}\left(\sqrt{16m^4 - 16m^2 + 16}t + x|\, m\right)^2 + \frac{1}{6}\sqrt{16m^4 - 16m^2 + 16} + \frac{1}{6}\sqrt{64m^4 - 64m^2 + 16}$.

- In particular, we determine the solutions of (7.1) corresponding to $m = 0$ and the soliton solutions corresponding to $m = 1$. Observing the conditions given in Solution 2 and Solution 3 which are $m > \frac{1}{\sqrt{2}} \vee m < -\frac{1}{\sqrt{2}}$ and $m > \frac{1}{\sqrt{2}} \vee m < -\frac{1}{\sqrt{2}}$ respectively, these solutions corresponding to (b) and (c) above are

  $u(x,t) = -2\mathrm{sech}^2(4t - x)$ and $u(x,t) = \frac{4}{3} - 2\mathrm{sech}^2(4t + x)$ respectively.

  The case for $m = 0$ in (a) resulted to $u(x,t) = -\frac{4}{3}$ and so no non-trivial solution was identified for this case.

The solutions of H in the form of u have physical significance. The soliton solution $u(x,t) = -2\mathrm{sech}^2(4t - x)$ in physical terms describes a trough of depth 2 traveling to the right with speed 4 and not changing its shape. On the other hand the soliton solution $u(x,t) = \frac{4}{3} - 2\mathrm{sech}^2(4t + x)$ in physical terms describes a trough of depth 2 traveling to the left with speed 4 and not changing its shape.

The plots corresponding to the traveling wave solutions of the KdV equation are depicted below.

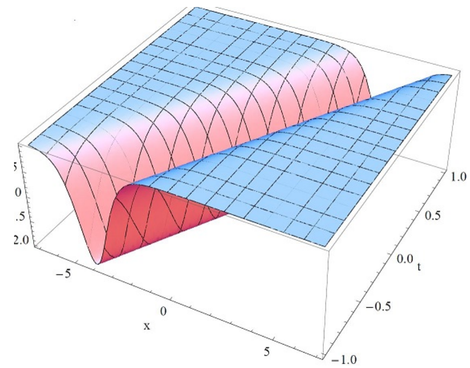


Figure 7.1: Plot depicting the KdV solution $u(x,t) = -2\text{sech}^2(4t - x)$ at $t = 0$.

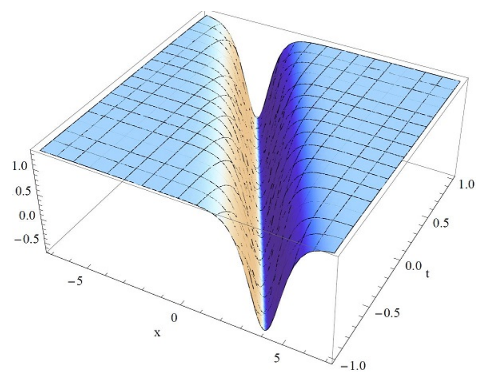


Figure 7.2: Plot depicting the KdV solution $u(x,t) = \frac{4}{3} - 2\text{sech}^2(4t + x)$ at $t = 0$.

7.2.2 Elliptic Weierstrass $\wp$ as a solution to the KdV equation

Another form of the KdV equation [15] equivalent to (7.1) is

$$u_t = \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}.$$

Similar to (7.1) this equation describes the time evolution of the wave as it travels in one direction. We look for a travelling wave solution to the KdV equation, of the form

$$u(x,t) = w(x + ct).$$

We get

$$cw' = \frac{3}{2}ww' + \frac{1}{4}w'''.$$

We integrate this equation to get

$$cw = \frac{3}{4}w^2 + \frac{1}{4}w'' + \gamma_1.$$

By multiplying the equation across by the integrating factor w' we get

$$cww' = \frac{3}{4}w^2w' + \frac{1}{4}w''w' + \gamma_1 w'.$$

A second integration gives

$$\frac{c}{2}w^2 = \frac{1}{4}w^3 + \frac{1}{8}(w')^2 + \gamma_1 w + \gamma_2.$$

Re-arranging the equation results to

$$(w')^2 = -2w^3 + 4cw^2 - 8\gamma_1 w - 8\gamma_2.$$

Now for any constant ω, a solution to this equation is given by

$$w(z) = -2\wp(z + \omega, \{g_2, g_3\}) + \frac{2}{3}c,$$

where

$$g_1 = \frac{4}{3}(c^2 - 3\gamma_1), \text{ and } g_2 = \frac{8c^3}{27} - \frac{4c\gamma_1}{3} - 2\gamma_2.$$

Thus,

$$u(x,t) = -2\wp(x + ct + \omega, g_2, g_3) + \frac{2c}{3}$$

is a solution to the KdV equation for any choice of ω, g_2, g_3 and c. A non-elliptic function solution of the KdV is of the form

$$u(x,t) = \frac{8k^2}{(e^{kx+k^3t} + e^{-kx-k^3t})^2}.$$

This particular solution describes for any k, a solitary wave that travels at speed k^2 and has a height of $2k^2$.

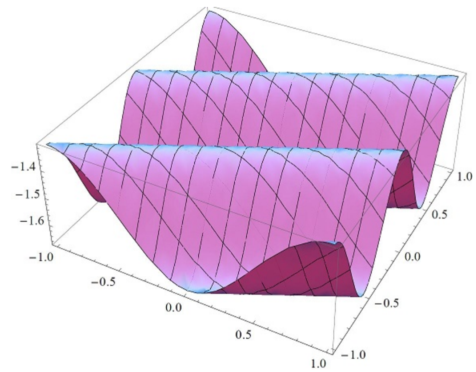


Figure 7.3: Plot depicting a Weierstrass $\wp$ solution for the KdV PDE

7.3 Riordan arrays determined from Elliptic cn^2

7.3.1 $\left[\mathbf{cn}(\xi,m)^2, \int \mathbf{cn}(\xi,m)^2 d\xi\right]$

The exponential Riordan array $\left[\text{cn}(\xi,m)^2, \int \text{cn}(\xi,m)^2 d\xi\right]$ belongs to the derivative subgroup (1.4.1).

$$\int \text{cn}(\xi,m)^2 d\xi = \frac{\left(\text{cn}(\xi|m)^2 + \frac{1}{m} - 1\right) E(\text{am}(\xi|m)|m)}{\text{dn}(\xi|m)\sqrt{1 - m\text{sn}(\xi|m)^2}} - \frac{\xi}{m} + \xi$$

with its Taylor series expansion given by

$$\xi - \frac{\xi^3}{3} + \frac{1}{15}(m+1)\xi^5 + \frac{1}{315}\left(-2m^2 - 13m - 2\right)\xi^7 + \frac{\left(m^3 + 30m^2 + 30m + 1\right)\xi^9}{2835} + O\left(\xi^{11}\right).$$

The coefficient array of $\left[\text{cn}(\xi,m)^2, \int \text{cn}(\xi,m)^2 d\xi\right]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -8 & 0 & 1 & 0 & 0 & 0 \\ 8(m+1) & 0 & -20 & 0 & 1 & 0 & 0 \\ 0 & 48m+88 & 0 & -40 & 0 & 1 & 0 \\ -16\left(2m^2+13m+2\right) & 0 & 56(3m+8) & 0 & -70 & 0 & 1 \end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 & 0 \\ 8(m-1) & 0 & -12 & 0 & 1 & 0 \\ 0 & 40(m-1) & 0 & -20 & 0 & 1 \\ -16(m-1)(2m-11) & 0 & 120(m-1) & 0 & -30 & 0 \end{pmatrix}$$

The production matrix of B in terms of $m = -1, 0, 1$ is given by

$$C = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 & 0 \\ -16 & 0 & -12 & 0 & 1 & 0 \\ 0 & -80 & 0 & -20 & 0 & 1 \\ -416 & 0 & -240 & 0 & -30 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 & 0 \\ -8 & 0 & -12 & 0 & 1 & 0 \\ 0 & -40 & 0 & -20 & 0 & 1 \\ -176 & 0 & -120 & 0 & -30 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 & 0 \\ 0 & 0 & -12 & 0 & 1 & 0 \\ 0 & 0 & 0 & -20 & 0 & 1 \\ 0 & 0 & 0 & 0 & -30 & 0 \end{pmatrix} \right\}.$$

Remark:

- The sequence $-2, -6, -12, -20, -30, ...$ positioned on the diagonal of the matrix C for $m = 1$ corresponds to **A002378** (Pronic Number) having the g.f

$$g(\xi) = \frac{2\xi}{(1-\xi)^3} \ \& \ g(n) = n(n+1).$$

Furthermore, the production matrix C for $m = 1$ is tridiagonal which implies that $\left[\text{cn}(\xi, m)^2, \int \text{cn}(\xi, m)^2 d\xi\right]^{-1}$ is the coefficient matrix of a family of formal orthogonal polynomials. For $m = 1$ we have

$$\left[\text{cn}(\xi, m)^2, \int \text{cn}(\xi, m)^2 d\xi\right] = \left[\text{sech}^2(\xi), -\cosh(\xi)\text{sech}(\xi)E\left(\frac{1}{2}\left(\pi - 4\tan^{-1}\left(e^z\right)\right)\middle| 1\right)\right].$$

$$\equiv \left[\text{sech}^2(\xi), \tanh(\xi)\right].$$

Furthermore,

$$\left[\text{sech}^2(\xi), \tanh(\xi)\right]^{-1} = \left[\frac{1}{1-\xi^2}, \tanh^{-1}(\xi)\right]$$

represents the coefficient matrix of the family of orthogonal polynomials. The three term recurrence relation for these polynomials is given by

$$P_{n+1}(\xi) = \xi P_n(\xi) + n(n+1)P_{n-1}(\xi)$$

with $P_0(\xi) = 1, \ P_1(\xi) = \xi$ s.t $-1 < \xi < 1$.

$$\text{In particular, let } Q_n(\xi) = \frac{P_n(i\xi)}{i^n}, \ (i^2 = -1)$$

we get

$$Q_{n+1}(\xi) = \xi Q_n(\xi) - n(n+1)Q_{n-1}(\xi), \ \forall n \geq 1.$$

- The first and the second generating function of $\left[\text{cn}(\xi, m)^2, \int \text{cn}(\xi, m)^2 d\xi\right]$ determine the solution of the KdV PDE.

7.3.2 $[\mathbf{cn}(z,m)^2, z]$

The exponential Riordan array $\left[\mathrm{cn}(z,m)^2, z\right]$ having coefficient matrix given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 \\ 8(m+1) & 0 & -12 & 0 & 1 \end{pmatrix}.$$

REMARK The numbers $2, 6, 12, 20, 30, ...$ correspond to the the OEIS number **A002378**

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 \\ 8m-4 & 0 & -6 & 0 \end{pmatrix}.$$

The production matrix of B for $m = -1, 0, 1$ is given by:

$$C = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 \\ -12 & 0 & -6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 \\ -4 & 0 & -6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 \\ 4 & 0 & -6 & 0 \end{pmatrix} \right\}.$$

The generating function of the c sequence of $\left[\mathrm{cn}(z,m)^2, z\right]$ in terms of m is

$$\frac{2\mathrm{dn}(z|m)\mathrm{sn}(z|m)}{\mathrm{cn}(z|m)}.$$

The generating function of the c sequence of $\left[\mathrm{cn}(z,m)^2, z\right]$ in terms of $m = 1$ and $m = 0$ are $2\tanh(z)$ and $2\tan(z)$ respectively.

7.3.3 $[\mathbf{cn}(z,m)^2, \mathbf{sn}(z,m)]$

The exponential Riordan array $\left[\mathrm{cn}(z,m)^2, \mathrm{sn}(z,m)\right]$ has its coefficient matrix given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -m-7 & 0 & 1 & 0 & 0 \\ 8(m+1) & 0 & -4(m+4) & 0 & 1 & 0 \\ 0 & m^2+74m+61 & 0 & -10(m+3) & 0 & 1 \end{pmatrix}.$$

Remark: The second column of the matrix A corresponds to the expansion of $\mathrm{cn}(z,m)^2\mathrm{sn}(z,m)$. The non-zero elements from the sequence generated by $\mathrm{cn}(z,m)^2\mathrm{sn}(z,m)$ forms a sequence of monic polynomials associated to the matrix in (5.2.7).

The production matrix corresponding to A is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -m-5 & 0 & 1 \\ 6(m-1) & 0 & -3(m+3) & 0 \end{pmatrix}.$$

The production matrices of B for $m = -1, 0, 1$ are as follows:

$$\left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 \\ -12 & 0 & -6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -5 & 0 & 1 \\ -6 & 0 & -9 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -6 & 0 & 1 \\ 0 & 0 & -12 & 0 \end{pmatrix} \right\}.$$

REMARK: The production matrix of $\left[\mathrm{cn}(z,m)^2, \mathrm{sn}(z,m)\right]$ is tri-diagonal for the case of $m = 1$. Therefore, the inverse of $\left[\mathrm{cn}(z,m)^2, \mathrm{sn}(z,m)\right]$ forms a family of monic orthogonal polynomials for $m = 1$. That is

$$\left[\mathrm{sech}^2(z), \tanh(z)\right]^{-1} = \left[\frac{1}{1-z^2}, \tanh^{-1}(z)\right]$$

represents the coefficient matrix of the family of orthogonal polynomials. The three term recurrence relation for these polynomials is given by

$$P_{n+1}(z) = zP_n(z) + n(n+1)P_{n-1}(z)$$

with $P_0(z) = 1,\ P_1(z) = z$ s.t $-1 < z < 1$.

$$\text{In particular, let } Q_n(z) = \frac{P_n(iz)}{i^n},\ (i^2 = -1)$$

we get

$$Q_{n+1}(z) = zQ_n(z) - n(n+1)Q_{n-1}(z),\ \forall n \geq 1.$$

7.4 The KdV Hierarchy

The equation representing the KdV hierarchy [56] is given by

$$U_{t_{2n+1}} + \partial_x \mathfrak{L}_{n+1}[U] = 0, \qquad n \geq 0$$

where $\mathfrak{L}$ satisfies $\partial_x \mathfrak{L}_{n+1}\{U\} = (\partial_{xxx} + 4U\partial_x + 2U_x)\,\mathfrak{L}_n\{U\}$ and

$$\mathfrak{L}_0 = \frac{1}{2} \qquad \mathfrak{L}_1\{U\} = U.$$

The first 3 terms of the infinite sequence of PDE's derived from the equation of the KdV hierarchy are listed as follows:

$$n = 0: \; U_{t_1} + U_x = 0 \qquad \mathfrak{L} = U$$

$$n = 1: \; U_{t_3} + 6UU_x + U_{xxx} = 0 \qquad \mathfrak{L}_2 = U_{xx} + 3U^2$$

$$n = 2: \; U_{t_5} + U_{5x} + 10UU_{xxx} + 20U_xU_{xxx} + 30U^2U_x = 0.$$

The travel wave solution is of the form $f(x \pm ct)$, where c is the speed of the waves. For the case $n = 0$, it can easily be verified that

$$f(x \pm t)$$

is a solution to $U_{t_1} + U_x = 0$.

The case for $n = 1$ is equivalent to the standard form of the KdV equation (7.1). It can be established in a similar procedure carried out in section (7.2.1) that the KdV hierarchy for $n \geq 2$ have solutions of the form $U(\xi) = a + bcn^2(\xi, m)$.

Chapter 8

Riordan arrays inspired by the analytical solutions of nonlinear PDE of wave propagation in nonlinear low pass electrical transmission lines

8.1 Introduction

In this chapter we establish the relationship between Riordan arrays and the solution to the nonlinear fourth order PDE which governs wave propagation in nonlinear low-pass electrical lines investigated in [124]. This nonlinear PDE is given by

$$\frac{\partial^2 V(x,t)}{\partial t^2} - \alpha\frac{\partial^2 V(x,t)}{\partial t^2} + \beta\frac{\partial^2 V^3(x,t)}{\partial t^2} - \delta^2\frac{\partial^2 V(x,t)}{\partial x^2} - \frac{\delta^4}{12}\frac{\partial^4 V(x,t)}{\partial x^4} \qquad (8.1)$$

where α, β, δ are constants, $V(x,t)$ is the voltage of the transmission lines such that x is the propagation distance and t is the slow time.

The proposed travel wave solutions of (8.1) in [124] is given by

$$V(\xi) = g_0 + \sum_{i=1}^{N} \left[\frac{z(\xi)}{1+z^2(\xi)}\right]^{i-1} \left\{g_i \left(\frac{z(\xi)}{1+z^2(\xi)}\right) + f_i \left(\frac{1-z^2(\xi)}{1+z^2(\xi)}\right)\right\} \quad (8.2)$$

where $z(\xi)$ satisfies

$$(z'(\xi))^2 = a + bz^2(\xi) + cz^4(\xi)$$

with the goal of determining the constants $a, b, c, g_0, g_i, f_i (i = 1, .., N)$ such that $g_N \neq 0$ or $f_N \neq 0$. Based on Kirchoff's law for the physical derivation of (8.1), N represents the total number of voltages measured.

In addition possible values of a, b, c in terms of the modulus m have been presented for 6 cases of Jacobi elliptic functions $z(\xi)$ which are

$$\mathrm{sn}(\xi), \mathrm{cn}(\xi), \mathrm{ns}(\xi), \mathrm{nc}(\xi), \mathrm{ns}(\xi) \pm \mathrm{cs}(\xi), \mathrm{nc}(\xi) \pm \mathrm{sc}(\xi).$$

In particular, for the case $z(\xi) = \mathrm{sn}(\xi, m)$ we determine that $V(\xi)$ is expressible in terms of a Riordan array.

Setting $j = i - 1$ in (8.2) and substituting $z(\xi) = \mathrm{sn}(\xi, m)$ gives

$$V(\xi) = g_0 + \sum_{j=0}^{N} \left[\frac{\mathrm{sn}(\xi, m)}{1+\mathrm{sn}^2(\xi, m)}\right]^{j} \left\{g_{j+1} \left(\frac{\mathrm{sn}(\xi, m)}{1+\mathrm{sn}^2(\xi, m)}\right) + f_{j+1} \left(\frac{1-\mathrm{sn}^2(\xi, m)}{1+\mathrm{sn}^2(\xi, m)}\right)\right\}. \quad (8.3)$$

Let $h(\xi) = \frac{\mathrm{sn}(\xi,m)}{1+\mathrm{sn}^2(\xi,m)}$.

On the other hand, let $d(\xi) = g_{j+1}\left(\frac{\mathrm{sn}(\xi,m)}{1+\mathrm{sn}^2(\xi,m)}\right) + f_{j+1}\left(\frac{1-\mathrm{sn}^2(\xi,m)}{1+\mathrm{sn}^2(\xi,m)}\right)$.

Rewriting equation (8.3) in terms of $d(\xi)$ and $h(\xi)$ results in

$$V(\xi) = g_0 + \sum_{j=0}^{N} d(\xi)h(\xi)^j. \quad (8.4)$$

Expanding the functions $h(\xi)$ and $d(\xi)$ respectively gives

$$\frac{\mathrm{sn}(\xi|m)}{\mathrm{sn}(\xi|m)^2+1} = \xi + \left(-\frac{m}{6} - \frac{7}{6}\right)\xi^3 + \frac{1}{120}\left(m^2 + 74m + 181\right)\xi^5 + \frac{\left(-m^3 - 681m^2 - 6939m - 9787\right)\xi^7}{5040} + O\left(\xi^9\right) \quad (8.5)$$

and

$$g_{j+1}\left(\frac{\operatorname{sn}(\xi,m)}{1+\operatorname{sn}^2(\xi,m)}\right)+f_{j+1}\left(\frac{1-\operatorname{sn}^2(\xi,m)}{1+\operatorname{sn}^2(\xi,m)}\right)=$$
$$f_{j+1}+\xi g_{j+1}+\xi^2\left(-2f_{j+1}-g_{j+1}\right)+\xi^3\left(\frac{5g_{j+1}}{6}-\frac{mg_{j+1}}{6}\right)+$$
$$\frac{1}{3}\xi^4\left(2mf_{j+1}+8f_{j+1}+mg_{j+1}-2g_{j+1}\right)+\frac{1}{120}\xi^5\left(m^2g_{j+1}-46mg_{j+1}+61g_{j+1}\right)+$$
$$\frac{1}{45}\xi^6\left(-4m^2f_{j+1}-86mf_{j+1}-154f_{j+1}-2m^2g_{j+1}+17mg_{j+1}-17g_{j+1}\right)+O\left(\xi^7\right) \tag{8.6}$$

Let $\hat{d}(\xi)=\frac{d(\xi)}{f_{j+1}}\in\mathcal{F}_0$ with $h(\xi)\in\mathcal{F}_1$ in (8.5). From the definition of the generic element of a Riordan array it follows that

$$d_{N,j}=[\xi^N]\frac{N!}{j!}\hat{d}(\xi)h(\xi)^j. \tag{8.7}$$

Rewriting (8.4) in terms of (8.7) gives

$$\begin{aligned}\hat{V}(\xi) &= g_0+\sum_{j=0}^{N}[\xi^N]\frac{N!}{j!}\hat{d}(\xi)h(\xi)^j\xi^j && (8.8)\\ &= g_0+\sum_{j=0}^{N}d_{N,j}\xi^j && (8.9)\\ &= g_0+\sum_{j=0}^{N}d_{N,j}[t^j]j!e^{t\xi}. && (8.10)\end{aligned}$$

That is

$$\hat{V}(\xi)=g_0+\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ \frac{g_{j+1}}{f_{j+1}} & 1 & 0 & 0 & 0 & \\ -\frac{2g_{j+1}}{f_{j+1}}-4 & \frac{2g_{j+1}}{f_{j+1}} & 1 & 0 & 0 & \\ -\frac{(m-5)g_{j+1}}{f_{j+1}} & -m-\frac{6g_{j+1}}{f_{j+1}}-19 & \frac{3g_{j+1}}{f_{j+1}} & 1 & 0 & \\ \frac{8(2(m+4)f_{j+1}+(m-2)g_{j+1})}{f_{j+1}} & -\frac{8(m+1)g_{j+1}}{f_{j+1}} & -4(m+13)-\frac{12g_{j+1}}{f_{j+1}} & \frac{4g_{j+1}}{f_{j+1}} & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\cdot\begin{pmatrix}1\\ \xi\\ \xi^2\\ \xi^3\\ \xi^4\\ \vdots\end{pmatrix}.$$

Thus,

$$V(\xi)=g_0+f_{j+1}\sum_{j=0}^{N}d_{N,j}\xi^j$$

can represent the family of solutions of the PDE (8.1) for the determined values of g_0, f_{j+1} and g_{j+1}.

Remark: The coefficients f_{j+1} and g_{j+1}. are determined by suitable methods and their values depend on the size given by the row number N for each case. The case for $z(\xi) = \text{ns}(\xi)$ can be used to express $V(\xi)$ in a similar procedure as above in terms of a Riordan array.

8.2 Generating functions from $\left(\frac{z(\xi)}{1+z^2(\xi)}\right)$ & $\left(\frac{1-z^2(\xi)}{1+z^2(\xi)}\right)$

We use the two quantities

$$\left(\frac{z(\xi)}{1+z^2(\xi)}\right) \& \left(\frac{1-z^2(\xi)}{1+z^2(\xi)}\right)$$

that define the solution $V(\xi)$ in (8.2) to show that for appropriate choice of $z(\xi)$ there exist some interesting Riordan arrays. Some of these Riordan arrays appear to be orthogonal for $m = 1$. The following sequences are generated below.

$$\frac{\text{sn}(\xi|m)}{\text{sn}(\xi|m)^2+1} = \xi + \left(-\frac{m}{6} - \frac{7}{6}\right)\xi^3 + \frac{1}{120}\left(m^2+74m+181\right)\xi^5 + \frac{\left(-m^3-681m^2-6939m-9787\right)\xi^7}{5040} + O\left(\xi^9\right) \quad (8.11)$$

$$\frac{\text{cn}(\xi|m)}{\text{cn}(\xi|m)^2+1} = \frac{1}{2} - \frac{\xi^4}{16} + \left(\frac{m}{24} - \frac{1}{48}\right)\xi^6 + \frac{\left(-16m^2+16m-1\right)\xi^8}{1280} + O\left(\xi^9\right) \quad (8.12)$$

$$\frac{1-\text{sn}(\xi|m)^2}{\text{sn}(\xi|m)^2+1} = 1 - 2\xi^2 + \frac{2}{3}(m+4)\xi^4 - \frac{2}{45}\left(2m^2+43m+77\right)\xi^6 + \frac{2}{315}\left(m^3+93m^2+597m+694\right)\xi^8 + O\left(\xi^9\right) \quad (8.13)$$

$$\frac{1-\text{cn}(\xi|m)^2}{\text{cn}(\xi|m)^2+1} = \frac{\xi^2}{2} + \left(\frac{1}{12} - \frac{m}{6}\right)\xi^4 + \frac{1}{360}\left(8m^2-8m-7\right)\xi^6 + \frac{\left(-8m^3+12m^2+138m-71\right)\xi^8}{5040} + O\left(\xi^9\right) \quad (8.14)$$

$$\frac{\text{ns}(\xi|m)}{\text{ns}(\xi|m)^2+1} = \xi + \left(-\frac{m}{6}-\frac{7}{6}\right)\xi^3 + \frac{1}{120}\left(m^2+74m+181\right)\xi^5 + \frac{\left(-m^3-681m^2-6939m-9787\right)\xi^7}{5040} + O\left(\xi^9\right) \quad (8.15)$$

$$\frac{\text{nc}(\xi|m)}{\text{nc}(\xi|m)^2+1} = \frac{1}{2} - \frac{\xi^4}{16} + \left(\frac{m}{24}-\frac{1}{48}\right)\xi^6 + \frac{\left(-16m^2+16m-1\right)\xi^8}{1280} + O\left(\xi^9\right) \quad (8.16)$$

$$\frac{1-\text{ns}(\xi|m)^2}{\text{ns}(\xi|m)^2+1} = -1 + 2\xi^2 - \frac{2}{3}(m+4)\xi^4 + \frac{2}{45}\left(2m^2+43m+77\right)\xi^6 - \frac{2}{315}\left(m^3+93m^2+597m+694\right)\xi^8 + O\left(\xi^9\right) \quad (8.17)$$

$$\frac{1-\text{nc}(\xi|m)^2}{\text{nc}(\xi|m)^2+1} = -\frac{\xi^2}{2} + \left(\frac{m}{6}-\frac{1}{12}\right)\xi^4 + \frac{1}{360}\left(-8m^2+8m+7\right)\xi^6 + \frac{\left(8m^3-12m^2-138m+71\right)\xi^8}{5040} + O\left(\xi^9\right) \quad (8.18)$$

The Riordan arrays below are derived from some of the sequences above.

8.3 $\left[1, \frac{\mathbf{sn}(\xi|m)}{\mathbf{sn}(\xi|m)^2+1}\right]$

The exponential Riordan array $\left[1, \frac{\text{sn}(\xi|m)}{\text{sn}(\xi|m)^2+1}\right]$ having coefficient matrix given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -m-7 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4(m+7) & 0 & 1 & 0 \\ 0 & m^2+74m+181 & 0 & -10(m+7) & 0 & 1 \end{pmatrix}.$$

Remark: The non-zero entries of the second column of A forms a triangle of numbers which has been treated in section (5.2.8).

The associated production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -m-7 & 0 & 1 & 0 \\ 0 & 0 & -3(m+7) & 0 & 1 \\ 0 & -3\left(m^2-6m+5\right) & 0 & -6(m+7) & 0 \end{pmatrix}.$$

For $m = -1, 0, 1$ respectively, $\left[1, \frac{\mathrm{sn}(\xi|m)}{\mathrm{sn}(\xi|m)^2+1}\right]$ evaluates to

$$\left\{\left[1, \frac{\mathrm{sn}(\xi|-1)}{\mathrm{sn}(\xi|-1)^2+1}\right], \left[1, \frac{\sin(\xi)}{\sin^2(\xi)+1}\right], \left[1, \frac{\tanh(\xi)}{\tanh^2(\xi)+1}\right]\right\}.$$

The production matrix of B for $m = -1, 0, 1$ are respectively given by

$$C = \left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 \\ 0 & 0 & -18 & 0 & 1 \\ 0 & -36 & 0 & -36 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -7 & 0 & 1 & 0 \\ 0 & 0 & -21 & 0 & 1 \\ 0 & -15 & 0 & -42 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -8 & 0 & 1 & 0 \\ 0 & 0 & -24 & 0 & 1 \\ 0 & 0 & 0 & -48 & 0 \end{pmatrix}\right\}.$$

8.4 $\left[2\frac{\mathbf{cn}(\xi|m)}{\mathbf{cn}(\xi|m)^2+1}, \xi\right]$

The exponential Riordan array $\left[2\frac{\mathrm{cn}(\xi|m)}{\mathrm{cn}(\xi|m)^2+1}, \xi\right]$ having as coefficient matrix A equals to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -15 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 30(2m-1) & 0 & -45 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 210(2m-1) & 0 & -105 & 0 & 0 & 0 & 1 & 0 \\ -63\left(16m^2-16m+1\right) & 0 & 840(2m-1) & 0 & -210 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The production matrix of A is

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -12 & 0 & 0 & 0 & 1 & 0 & 0 \\ 30(2m-1) & 0 & -30 & 0 & 0 & 0 & 1 & 0 \\ 0 & 180(2m-1) & 0 & -60 & 0 & 0 & 0 & 1 \\ -126\left(8m^2-8m+3\right) & 0 & 630(2m-1) & 0 & -105 & 0 & 0 & 0 \end{pmatrix}.$$

The production matrices of B for $m = -1, 0, 1$ respectively are

$$C = \left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -12 & 0 & 0 & 0 & 1 & 0 \\ -90 & 0 & -30 & 0 & 0 & 0 & 1 \\ 0 & -540 & 0 & -60 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -12 & 0 & 0 & 0 & 1 & 0 \\ -30 & 0 & -30 & 0 & 0 & 0 & 1 \\ 0 & -180 & 0 & -60 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -12 & 0 & 0 & 0 & 1 & 0 \\ 30 & 0 & -30 & 0 & 0 & 0 & 1 \\ 0 & 180 & 0 & -60 & 0 & 0 & 0 \end{pmatrix}\right\}.$$

8.5 $\left[\frac{1-\mathbf{sn}(\xi|m)^2}{\mathbf{sn}(\xi|m)^2+1}, \xi\right]$

The exponential Riordan array $\left[\frac{1-\text{sn}(\xi|m)^2}{\text{sn}(\xi|m)^2+1}, \xi\right]$ having coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 & 0 \\ 0 & -12 & 0 & 1 & 0 & 0 \\ 16(m+4) & 0 & -24 & 0 & 1 & 0 \\ 0 & 80(m+4) & 0 & -40 & 0 & 1 \end{pmatrix}.$$

The production matrix of A is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -8 & 0 & 1 & 0 \\ 16(m+1) & 0 & -12 & 0 & 1 \\ 0 & 64(m+1) & 0 & -16 & 0 \end{pmatrix}$$

For $m = -1, 0, 1$ respectively, $\left[\frac{1-\text{sn}(\xi|m)^2}{\text{sn}(\xi|m)^2+1}, \xi\right]$ evaluates to

$$\left\{\left[\frac{1-\text{sn}(\xi|-1)^2}{\text{sn}(\xi|-1)^2+1}, \xi\right], \left[\frac{1-\sin^2(\xi)}{\sin^2(\xi)+1}, \xi\right], \left[\frac{1-\tanh^2(\xi)}{\tanh^2(\xi)+1}, \xi\right]\right\}.$$

The production matrices of B for $m = -1, 0, 1$ are given by

$$C = \left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -8 & 0 & 1 & 0 \\ 0 & 0 & -12 & 0 & 1 \\ 0 & 0 & 0 & -16 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -8 & 0 & 1 & 0 \\ 16 & 0 & -12 & 0 & 1 \\ 0 & 64 & 0 & -16 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -8 & 0 & 1 & 0 \\ 32 & 0 & -12 & 0 & 1 \\ 0 & 128 & 0 & -16 & 0 \end{pmatrix}\right\}.$$

8.6 $\left[1, \frac{\mathbf{ns}(\xi|m)}{\mathbf{ns}(\xi|m)^2+1}\right]$

Consider the exponential Riordan array $\left[1, \frac{\text{ns}(\xi|m)}{\text{ns}(\xi|m)^2+1}\right]$ which has the coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -m-7 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4(m+7) & 0 & 1 & 0 \\ 0 & m^2+74m+181 & 0 & -10(m+7) & 0 & 1 \end{pmatrix}.$$

The production matrix of A is

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -m-7 & 0 & 1 & 0 \\ 0 & 0 & -3(m+7) & 0 & 1 \\ 0 & -3\left(m^2-6m+5\right) & 0 & -6(m+7) & 0 \end{pmatrix}.$$

For $m = -1, 0, 1$ respectively, $\left[1, \frac{\text{ns}(\xi|m)}{\text{ns}(\xi|m)^2+1}\right]$ evaluates to

$$\left\{\left[1, \frac{\text{ns}(\xi|-1)}{\text{ns}(\xi|-1)^2+1}\right], \left[1, \frac{\csc(\xi)}{\csc^2(\xi)+1}\right], \left[1, \frac{\coth(\xi)}{\coth^2(\xi)+1}\right]\right\}.$$

The production matrix of B for the case $m = -1, 0, 1$ respectively are given by

$$C = \left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 \\ 0 & 0 & -18 & 0 & 1 \\ 0 & -36 & 0 & -36 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -7 & 0 & 1 & 0 \\ 0 & 0 & -21 & 0 & 1 \\ 0 & -15 & 0 & -42 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -8 & 0 & 1 & 0 \\ 0 & 0 & -24 & 0 & 1 \\ 0 & 0 & 0 & -48 & 0 \end{pmatrix}\right\}.$$

8.7 Riordan arrays from $\frac{\textbf{cn}(\xi|m)}{1-\textbf{sn}(\xi|m)}$ and $\frac{\textbf{cn}(\xi|m)}{1+\textbf{sn}(\xi|m)}$

Consider the series

$$\frac{\text{cn}(\xi|m)}{1-\text{sn}(\xi|m)} = 1+\xi+\frac{\xi^2}{2}+\left(\frac{1}{3}-\frac{m}{6}\right)\xi^3+\left(\frac{5}{24}-\frac{m}{6}\right)\xi^4+\frac{1}{120}\left(m^2-16m+16\right)\xi^5+ \frac{1}{720}\left(16m^2-76m+61\right)\xi^6+\frac{\left(-m^3+138m^2-408m+272\right)\xi^7}{5040}+ \frac{\left(-64m^3+1104m^2-2424m+1385\right)\xi^8}{40320}+O\left(\xi^9\right) \quad (8.19)$$

$$\text{nc}(\xi|m)+\text{sc}(\xi|m) = 1+\xi+\frac{\xi^2}{2}+\left(\frac{1}{3}-\frac{m}{6}\right)\xi^3+\left(\frac{5}{24}-\frac{m}{6}\right)\xi^4+\frac{1}{120}\left(m^2-16m+16\right)\xi^5+ \frac{1}{720}\left(16m^2-76m+61\right)\xi^6+\frac{\left(-m^3+138m^2-408m+272\right)\xi^7}{5040}+ \frac{\left(-64m^3+1104m^2-2424m+1385\right)\xi^8}{40320}+O\left(\xi^9\right) \quad (8.20)$$

$$\frac{\mathrm{cn}(\xi|m)}{\mathrm{sn}(\xi|m)+1} = 1-\xi+\frac{\xi^2}{2}+\frac{1}{6}(m-2)\xi^3+\left(\frac{5}{24}-\frac{m}{6}\right)\xi^4+\frac{1}{120}\left(-m^2+16m-16\right)\xi^5+ \frac{1}{720}\left(16m^2-76m+61\right)\xi^6+\frac{\left(m^3-138m^2+408m-272\right)\xi^7}{5040}+ \frac{\left(-64m^3+1104m^2-2424m+1385\right)\xi^8}{40320}+O\left(\xi^9\right) \quad (8.21)$$

$$\mathrm{nc}(\xi|m)-\mathrm{sc}(\xi|m) = 1-\xi+\frac{\xi^2}{2}+\frac{1}{6}(m-2)\xi^3+\left(\frac{5}{24}-\frac{m}{6}\right)\xi^4+\frac{1}{120}\left(-m^2+16m-16\right)\xi^5+ \frac{1}{720}\left(16m^2-76m+61\right)\xi^6+O\left(\xi^7\right) \quad (8.22)$$

REMARK: Equation (8.19) is equivalent to equation (8.20) and similarly equation (8.21) is equivalent to equation (8.22). That is

$$\frac{\mathrm{cn}(\xi|m)}{1-\mathrm{sn}(\xi|m)} \equiv \mathrm{nc}(\xi|m)+\mathrm{sc}(\xi|m) \ \& \ \frac{\mathrm{cn}(\xi|m)}{1+\mathrm{sn}(\xi|m)} \equiv \mathrm{nc}(\xi|m)-\mathrm{sc}(\xi|m). \quad (8.23)$$

8.7.1 $\left[\frac{\mathbf{cn}(\xi|m)}{1+\mathbf{sn}(\xi|m)}, \xi\right]$

Consider the Riordan array of the Appell subgroup $\left[\frac{\mathrm{cn}(\xi|m)}{1+\mathrm{sn}(\xi|m)}, \xi\right]$ having as coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ m-2 & 3 & -3 & 1 & 0 & 0 \\ 5-4m & 4(m-2) & 6 & -4 & 1 & 0 \\ -m^2+16m-16 & 25-20m & 10(m-2) & 10 & -5 & 1 \end{pmatrix}.$$

The production matrix of A is given by

$$B = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ m-1 & 0 & -1 & 1 & 0 \\ 0 & 3m-3 & 0 & -1 & 1 \\ -m^2+6m-5 & 0 & 6m-6 & 0 & -1 \end{pmatrix}.$$

The production matrix of B for the case $m = -1, 0, 1$ are given by

$$C = \left\{ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -2 & 0 & -1 & 1 & 0 \\ 0 & -6 & 0 & -1 & 1 \\ -12 & 0 & -12 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ 0 & -3 & 0 & -1 & 1 \\ -5 & 0 & -6 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \right\}.$$

Remark: The production matrix in C for the case $m = 1$ corresponds to the production matrix of the inverse Pascal triangle.

8.7.1.1 The Inverse of $\left[\frac{\mathbf{cn}(\xi|m)}{1+\mathbf{sn}(\xi|m)}, \xi\right]$

By applying (1.4) , the inverse of the matrix A corresponds to the Riordan matrix $\left[\frac{\mathrm{sn}(\xi|m)+1}{\mathrm{cn}(\xi|m)}, \xi\right]$ which has coefficient matrix given by

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 2-m & 3 & 3 & 1 & 0 & 0 \\ 5-4m & 8-4m & 6 & 4 & 1 & 0 \\ m^2-16m+16 & 25-20m & -10(m-2) & 10 & 5 & 1 \end{pmatrix}.$$

The production matrix of D in terms of m is given by

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1-m & 0 & 1 & 1 & 0 \\ 0 & 3-3m & 0 & 1 & 1 \\ m^2-6m+5 & 0 & 6-6m & 0 & 1 \end{pmatrix}.$$

The production matrix E for the case $m = -1, 0, 1$ are as follows

$$F = \left\{ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 6 & 0 & 1 & 1 \\ 12 & 0 & 12 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 & 1 \\ 5 & 0 & 6 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

REMARK 1: The production matrix derived from D for the case $m = 1$ in F corresponds to the production matrix of the Pascal triangle. The Riordan matrix for the case $m = 1$ in D corresponds to $[\cosh(\xi)(\tanh(\xi) + 1), \xi]$. Furthermore, we show that $\cosh(z)(\tanh(z) + 1) \equiv e^z$ using the relations

$$\cosh(z) = \frac{e^z + e^{-z}}{2} \text{ and } \tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

That is

$$\begin{aligned} \cosh(z)(\tanh(z) + 1) &= \frac{e^z + e^{-z}}{2}\left(1 + \frac{e^z - e^{-z}}{e^z + e^{-z}}\right) \\ &= \frac{e^z + e^{-z}}{2} + \frac{e^z - e^{-z}}{2} \\ &= e^z. \end{aligned}$$

Thus, $[\cosh(z)(\tanh(z)+1), z]$ defined by hyperbolic functions generates the Pascal triangle in addition to the already known Riordan arrays $\left(\frac{1}{1-z}, \frac{z}{1-z}\right)$ and $[e^z, z]$.

Remark 2: By extracting the monic polynomials located on the odd positions of the expanded matrix D starting from the first row $n=0$, results to the submatrix given by

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & -16 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 272 & -408 & 138 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7936 & -15872 & 9168 & -1232 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 353792 & -884480 & 729728 & -210112 & 11074 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark:

- The row sums of the first column of F is given by $(1,1,1,1,1,1,1,..)$ which corresponds to **A000012** having e.g.f e^z
- The numbers $1, 2, 16, 272, 7936, 353792, ...$ located on the first column of F correspond to **A000182** having the e.g.f $\tanh(z)$.

Using the generating function for the row sums of F given by e^z and the generating function of the first column of F given by $\tanh(z)$ we determine a new Riordan array from these results. Recall that for an exponential Riordan array $[g(z), f(z)]$, the row sums is given by

$$g(z)e^{f(z)}$$

where $g(z) \in \mathcal{F}_0$ and $f(z) \in \mathcal{F}_1$.

But $\tanh(z) \in \mathcal{F}_1$ since

$$\tanh(z) = z - \frac{z^3}{3} + \frac{2z^5}{15} - \frac{17z^7}{315} + O\left(z^9\right).$$

The first generating function of the new Riordan array can be determined as follows:

$$\begin{aligned} e^z &= g(z)e^{\tanh(z)} \\ g(z) &= \frac{e^z}{e^{\tanh(z)}} \\ &= e^{z-\tanh(z)}. \end{aligned}$$

That is $g(z) = e^{z-\tanh(z)} \in \mathcal{F}_0$ since

$$e^{z-\tanh(z)} = 1 + \frac{z^3}{3} - \frac{2z^5}{15} + \frac{z^6}{18} + \frac{17z^7}{315} - \frac{2z^8}{45} + O\left(z^9\right).$$

The coefficient matrix of the exponential Riordan array

$$\left[e^{z-\tanh(z)}, \tanh(z)\right]$$

is given by

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & -8 & 0 & 1 & 0 & 0 & 0 & 0 \\ -16 & 16 & 20 & -20 & 0 & 1 & 0 & 0 & 0 \\ 40 & -176 & 136 & 40 & -40 & 0 & 1 & 0 & 0 \\ 272 & 8 & -896 & 616 & 70 & -70 & 0 & 1 & 0 \\ -1792 & 5760 & -2848 & -3136 & 2016 & 112 & -112 & 0 & 1 \end{pmatrix}.$$

The production matrix of G is given by

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & -6 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 12 & -12 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 20 & -20 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 30 & -30 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 42 & -42 & 0 \end{pmatrix}.$$

Remark: The 4-diagonal production matrix H indicates that G^{-1} is the coefficient array of a family of 2-orthogonal polynomials [61].

On the other hand by multiplying the matrix F with $(-1)^n$ where n is the column number starting from 0, gives the submatrix

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 16 & 16 & 1 & 0 & 0 & 0 & 0 \\ 272 & 408 & 138 & 1 & 0 & 0 & 0 \\ 7936 & 15872 & 9168 & 1232 & 1 & 0 & 0 \\ 353792 & 884480 & 729728 & 210112 & 11074 & 1 & 0 \end{pmatrix}.$$

But in the case of I no significant results were determined for its row sums.

8.8 $\left[\frac{1-z^2(\xi)}{1+z^2(\xi)}, \frac{z(\xi)}{1+z^2(\xi)}\right]$

In this section we consider the Riordan array $\left[\frac{1-z^2(\xi)}{1+z^2(\xi)}, \frac{z(\xi)}{1+z^2(\xi)}\right]$ which corresponds to the two factors of the proposed travel wave solutions (8.2) for the case $z(\xi) = \mathrm{sn}(\xi|m)$ and $z(\xi) = \mathrm{ns}(\xi|m)$.

8.8.1 Case 1: $z(\xi) = \mathbf{sn}(\xi|m)$

If $z(\xi) = \mathrm{sn}(\xi, m)$ we have the sequences:

$$\frac{\mathrm{sn}(\xi,m)}{1+\mathrm{sn}^2(\xi,m)} = \xi + \left(-\frac{m}{6} - \frac{7}{6}\right)\xi^3 + \frac{1}{120}\left(m^2 + 74m + 181\right)\xi^5 + \frac{\left(-m^3 - 681m^2 - 6939m - 9787\right)\xi^7}{5040} + O\left(\xi^9\right) \quad (8.24)$$

$$\frac{1-\mathtt{sn}^2(\xi,m)}{1+\mathrm{sn}^2(\xi,m)} = 1 - 2\xi^2 + \frac{2}{3}(m+4)\xi^4 - \frac{2}{45}\left(2m^2 + 43m + 77\right)\xi^6 + \frac{2}{315}\left(m^3 + 93m^2 + 597m + 694\right)\xi^8 + O\left(\xi^9\right) \quad (8.25)$$

The Riordan array $\left[\frac{1-\mathrm{sn}^2(\xi,m)}{1+\mathrm{sn}^2(\xi,m)}, \frac{\mathrm{sn}(\xi,m)}{1+\mathrm{sn}^2(\xi,m)}\right]$ has the coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 & 0 \\ 0 & -m-19 & 0 & 1 & 0 & 0 \\ 16(m+4) & 0 & -4(m+13) & 0 & 1 & 0 \\ 0 & m^2+194m+781 & 0 & -10(m+11) & 0 & 1 \end{pmatrix}.$$

Remark The non-zero entries of the second column of A corresponds to the triangle of numbers

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -19 & -1 & 0 & 0 & 0 \\ 781 & 194 & 1 & 0 & 0 \\ -57919 & -28947 & -1773 & -1 & 0 \\ 6823801 & 5269828 & 823758 & 15988 & 1 \end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -m-15 & 0 & 1 & 0 \\ 12(m-1) & 0 & -3(m+11) & 0 & 1 \\ 0 & -3\left(m^2-22m+21\right) & 0 & -6m-58 & 0 \end{pmatrix}.$$

The production matrix B for the case $m=-1,0,1$ is given by

$$C=\left\{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -14 & 0 & 1 & 0 \\ -24 & 0 & -30 & 0 & 1 \\ 0 & -132 & 0 & -52 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -15 & 0 & 1 & 0 \\ -12 & 0 & -33 & 0 & 1 \\ 0 & -63 & 0 & -58 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -16 & 0 & 1 & 0 \\ 0 & 0 & -36 & 0 & 1 \\ 0 & 0 & 0 & -64 & 0 \end{pmatrix}\right\}.$$

REMARK: The tri-diagonal nature of the production matrix C for the case $m=1$ shows that the inverse Riordan array of A corresponds to the coefficient matrix of a family of formal orthogonal polynomial sequences. This gives some indication to the possible relationship between orthogonal polynomials and the solution of the travel wave solution (8.2). For $m=1$, $\left[\frac{1+\tanh^2(\xi)}{1-\tanh^2(\xi)},\frac{\tanh(\xi)}{1-\tanh^2(\xi)}\right]^{-1}=$

$$\left[\frac{1}{\sqrt{1-4\xi^2}},\frac{1}{4}\log\left(\frac{1+2\xi}{1-2\xi}\right)\right].$$

This is the coefficient array of the orthogonal polynomials. The three-term recurrence relation representing these polynomials is given by

$$P_n(\xi)=\xi P_{n-1}(\xi)+4(n-1)^2P_{n-2}(\xi)$$

with $P_0(\xi)=1,\ P_1(\xi)=\xi$.

In particular, let $Q_{n-1}(\xi)=\dfrac{P_{n-1}(i\xi)}{i^n},\ (i^2=-1)$

we get

$$Q_n(\xi)=\xi Q_{n-1}(\xi)-4(n-1)^2Q_{n-2}(\xi),\ \forall n\geq 1.$$

8.8.1.1 The Inverse of $\left[\frac{1-\mathbf{sn}^2(\xi,m)}{1+\mathbf{sn}^2(\xi,m)},\frac{\mathbf{sn}(\xi,m)}{1+\mathbf{sn}^2(\xi,m)}\right]$

For a Riordan array $[g,f]$, the inverse $[g,f]^{-1}=\left[\frac{1}{g(\bar{f})},\bar{f}\right]$.

Using $[g,f]=\left[\frac{1-\text{sn}^2(\xi,m)}{1+\text{sn}^2(\xi,m)},\frac{\text{sn}(\xi,m)}{1+\text{sn}^2(\xi,m)}\right]$,

$$\bar{f}=\text{sn}^{-1}\left(\frac{1-\sqrt{1-4\xi^2}}{2\xi}\middle| m\right)$$

$$\frac{1}{g(\bar{f})}=\frac{1-\sqrt{1-4\xi^2}}{4\xi^2+\sqrt{1-4\xi^2}-1}.$$

Recall the generating function of the Catalan numbers **A000108** is given by

$$C(x)=\frac{1-\sqrt{1-4x}}{2x}.$$

Therefore,

$$\bar{f}=\text{sn}^{-1}\left(\xi C(\xi^2),m\right)$$

where $C(\xi^2)$ is the generating function of the aerated Catalan numbers **A126120.**

On the other hand

$$\frac{1}{g(\bar{f})}=\frac{1-\sqrt{1-4\xi^2}}{4\xi^2+\sqrt{1-4\xi^2}-1}=\frac{1}{\sqrt{1-4z^2}},$$

is the generating function corresponding to **A126869** which are the central binomial coefficients **A000984** interpolated with $0's$.

The coefficient array of $\left[\frac{1}{\sqrt{1-4z^2}},\text{sn}^{-1}\left(\frac{1-\sqrt{1-4\xi^2}}{2\xi}\middle| m\right)\right]$ which corresponds to the inverse of $\left[\frac{1-\text{sn}^2(\xi,m)}{1+\text{sn}^2(\xi,m)},\frac{\text{sn}(\xi,m)}{1+\text{sn}^2(\xi,m)}\right]$ is given by

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 4 & 0 & 1 & 0 & 0 & 0\\ 0 & m+19 & 0 & 1 & 0 & 0\\ 144 & 0 & 4m+52 & 0 & 1 & 0\\ 0 & 9m^2+106m+1309 & 0 & 10m+110 & 0 & 1 \end{pmatrix}.$$

The corresponding production matrix of A in terms of m is given by

$$B=\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 4 & 0 & 1 & 0 & 0\\ 0 & m+15 & 0 & 1 & 0\\ 68-4m & 0 & 3m+33 & 0 & 1\\ 0 & 5m^2-6m+385 & 0 & 6m+58 & 0 \end{pmatrix}.$$

8.8.2 Case 2: $z(\xi) = \mathbf{ns}(\xi, m)$

If $z(\xi) = \text{ns}(\xi|m)$ we have the sequences:

$$\frac{\text{ns}(\xi,m)}{1+\text{ns}^2(\xi,m)} = \xi + \left(-\frac{m}{6} - \frac{7}{6}\right)\xi^3 + \frac{1}{120}\left(m^2+74m+181\right)\xi^5 + \frac{\left(-m^3-681m^2-6939m-9787\right)\xi^7}{5040} + O\left(\xi^9\right) \quad (8.26)$$

$$\frac{1-\text{ns}^2(\xi,m)}{1+\text{ns}^2(\xi,m)} = -1 + 2\xi^2 - \frac{2}{3}(m+4)\xi^4 + \frac{2}{45}\left(2m^2+43m+77\right)\xi^6 - \frac{2}{315}\left(m^3+93m^2+597m+694\right)\xi^8 + O\left(\xi^9\right) \quad (8.27)$$

The Riordan array $\left(\frac{1-\text{ns}^2(\xi,m)}{1+\text{ns}^2(\xi,m)}, \frac{\text{ns}(\xi,m)}{1+\text{ns}^2(\xi,m)}\right)$ has coefficient matrix given by

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 4 & 0 & -1 & 0 & 0 & 0 \\ 0 & m+19 & 0 & -1 & 0 & 0 \\ -16(m+4) & 0 & 4(m+13) & 0 & -1 & 0 \\ 0 & -m^2-194m-781 & 0 & 10(m+11) & 0 & -1 \end{pmatrix}.$$

The production matrix of A in terms of m is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -m-15 & 0 & 1 & 0 \\ 12(m-1) & 0 & -3(m+11) & 0 & 1 \\ 0 & -3\left(m^2-22m+21\right) & 0 & -6m-58 & 0 \end{pmatrix}.$$

The production matrix of B in terms of $m = -1, 0, 1$ is given by

$$C = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -14 & 0 & 1 & 0 \\ -24 & 0 & -30 & 0 & 1 \\ 0 & -132 & 0 & -52 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -15 & 0 & 1 & 0 \\ -12 & 0 & -33 & 0 & 1 \\ 0 & -63 & 0 & -58 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & -16 & 0 & 1 & 0 \\ 0 & 0 & -36 & 0 & 1 \\ 0 & 0 & 0 & -64 & 0 \end{pmatrix} \right\}.$$

REMARK: The tri-diagonal nature of the production matrix C for the case $m = 1$ shows that the inverse Riordan array of A corresponds to the coefficient matrix of a family of formal orthogonal polynomial sequences. This gives a possible indication of the relationship between orthogonal polynomials and the solution of the travel wave solution (8.2). For $m = 1$, $\left[\frac{1+\tanh^2(\xi)}{1-\tanh^2(\xi)}, \frac{\tanh(\xi)}{1-\tanh^2(\xi)}\right]^{-1} =$

$$\left[\frac{1}{\sqrt{1-4\xi^2}}, \frac{1}{4}\log\left(\frac{1+2\xi}{1-2\xi}\right)\right].$$

This is the coefficient array of the orthogonal polynomials. The three-term recurrence relation representing these polynomials is given by

$$P_n(\xi) = \xi P_{n-1}(\xi) + 4(n-1)^2 P_{n-2}(\xi)$$

with $P_0(\xi) = 1,\ P_1(\xi) = \xi$.

In particular, let $Q_{n-1}(\xi) = \dfrac{P_{n-1}(i\xi)}{i^n},\ (i^2 = -1)$

we get

$$Q_n(\xi) = \xi Q_{n-1}(\xi) - 4(n-1)^2 Q_{n-2}(\xi),\ \forall n \geq 1.$$

Chapter 9

Riordan arrays related to the FRLW Cosmological model

9.1 Introduction

Previously we studied elliptic functions representing the exact solutions to wave-like problems in water waves modeled by KdV (7) and low pass electrical transmission (8). In this chapter, we examine some other forms of elliptic functions in order to determine the Riordan array that represent exact solutions to the Einstein gravitational fields equations present in the FRLW(Friedmann-Robertson-Lemaître-Walker) cosmolgy. The FRLW cosmological model mathematically describes the expansion of the universe by assuming in particular that our current expanding universe is on a large scale homogeneous and isotropic. In essence, it provides the solutions to the Einstein field equations. The FRLW assumption on a d- dimensional spacetime is given by

$$ds^2 = -dt^2 + \tilde{a}(t)^2 \left(\frac{dr^2}{1 - k'r^2} + r^2 d\Omega_{d-2}^2 \right) \tag{9.1}$$

where $\tilde{a}(t)$ is the cosmic scale factor and $k' \in \{-1, 0, 1\}$ is the curvature parameter. On the other hand, the Einstein's equations associated to (9.1) is given by

$$\frac{(d-1)(d-2)}{2}\left(H^2+\frac{k'}{\tilde{a}^2}\right)=\kappa_d\rho(t)+\Lambda \tag{9.2}$$

where $\rho(t)$ is the pressure of the fluid, $\Lambda > 0$ is the cosmological constant and $\Lambda_d = 8\pi G_d$, where G_d is a generalization of Newton's constant to $d-$ dimensional spacetime, $H(t) = def\ \dot{\tilde{a}}/\tilde{a}(t)$. Further details on the Einstein field equations is given in [34]. In particular, the elliptic functions determining the solution to the Einstein gravitational equations which govern the evolution of the universe over time has been treated in the paper [34]. These solutions are given in the form below:

$$a_{\mathrm{sn}}(\eta)=\frac{k}{\sqrt{\tilde{\Lambda}(1+k^2)}}sn\left(\frac{\eta}{\sqrt{1+k^2}},k\right) \tag{9.3}$$

$$a_{\mathrm{ns}}(\eta)=\frac{k}{\sqrt{\tilde{\Lambda}(1+k^2)}}ns\left(\frac{\eta}{\sqrt{1+k^2}},k\right) \tag{9.4}$$

$$a_{\mathrm{cd}}(\eta)=\frac{k}{\sqrt{\tilde{\Lambda}(1+k^2)}}cd\left(\frac{\eta}{\sqrt{1+k^2}},k\right) \tag{9.5}$$

$$a_{\mathrm{dc}}(\eta)=\frac{k}{\sqrt{\tilde{\Lambda}(1+k^2)}}dc\left(\frac{\eta}{\sqrt{1+k^2}},k\right) \tag{9.6}$$

Using the elliptic function given above we shall construct proper Riordan arrays belonging to the Appell subgroup that represent these solutions.

9.2 $\left[\frac{k}{\sqrt{\tilde{\Lambda}(1+k^2)}}\mathbf{cd}\left(\frac{\eta}{\sqrt{1+k^2}},k\right),\eta\right]\equiv[a_{\mathbf{cd}}(\eta),\eta]$

In particular,

$$a_{cd}(\eta)=\frac{k}{\sqrt{(k^2+1)\Lambda}}+\frac{\eta^2(k-1)k}{2(k^2+1)\sqrt{(k^2+1)\Lambda}}+\frac{\eta^4k\left(5k^2-6k+1\right)}{24(k^2+1)^2\sqrt{(k^2+1)\Lambda}}+\frac{\eta^6k\left(61k^3-107k^2+47k-1\right)}{720(k^2+1)^3\sqrt{(k^2+1)\Lambda}}+\frac{\eta^8k\left(1385k^4-3116k^3+2142k^2-412k+1\right)}{40320(k^2+1)^4\sqrt{(k^2+1)\Lambda}}+O\left(\eta^9\right)$$

We can construct the proper exponential Riordan array $\left[\frac{k}{\sqrt{\tilde{\Lambda}(1+k^2)}}\mathrm{cd}\left(\frac{\eta}{\sqrt{1+k^2}},k\right),\eta\right]\equiv$

$[a_{\mathrm{cd}}(\eta), \eta]$ having coefficient matrix given by

$$A = \begin{pmatrix} \frac{1}{\sqrt{(k^2+1)\Lambda}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{(k^2+1)\Lambda}} & 0 & 0 & 0 & 0 \\ \frac{(k-1)\Lambda}{((k^2+1)\Lambda)^{3/2}} & 0 & \frac{1}{\sqrt{(k^2+1)\Lambda}} & 0 & 0 & 0 \\ 0 & \frac{3(k-1)\Lambda}{((k^2+1)\Lambda)^{3/2}} & 0 & \frac{1}{\sqrt{(k^2+1)\Lambda}} & 0 & 0 \\ \frac{(k-1)(5k-1)}{(k^2+1)^2\sqrt{(k^2+1)\Lambda}} & 0 & \frac{6(k-1)\Lambda}{((k^2+1)\Lambda)^{3/2}} & 0 & \frac{1}{\sqrt{(k^2+1)\Lambda}} & 0 \\ 0 & \frac{5(k-1)(5k-1)}{(k^2+1)^2\sqrt{(k^2+1)\Lambda}} & 0 & \frac{10(k-1)\Lambda}{((k^2+1)\Lambda)^{3/2}} & 0 & \frac{1}{\sqrt{(k^2+1)\Lambda}} \end{pmatrix}.$$

The production matrix of A in terms of k where $0 < k < 1$ which corresponds to non elementary functions is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{k-1}{k^2+1} & 0 & 1 & 0 & 0 \\ 0 & \frac{2(k-1)}{k^2+1} & 0 & 1 & 0 \\ \frac{2(k^2-1)}{(k^2+1)^2} & 0 & \frac{3(k-1)}{k^2+1} & 0 & 1 \\ 0 & \frac{8(k^2-1)}{(k^2+1)^2} & 0 & \frac{4(k-1)}{k^2+1} & 0 \end{pmatrix}.$$

The production matrix B for the case $k = -1, 0, 1$ corresponding to the elementary trigonometric function $\cos(\eta)$ and the identity matrix $[1, \eta]$

$$C = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & -4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ -2 & 0 & -3 & 0 & 1 \\ 0 & -8 & 0 & -4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

The generating function of the c sequence of the production matrix of $[a_{cd}(\eta), \eta]$ in terms of modulus k is given by

$$\frac{(k-1)\mathrm{nd}\left(\frac{\eta}{\sqrt{k^2+1}}\middle| k\right)\mathrm{sd}\left(\frac{\eta}{\sqrt{k^2+1}}\middle| k\right)}{\sqrt{k^2+1}\,\mathrm{cd}\left(\frac{\eta}{\sqrt{k^2+1}}\middle| k\right)}. \tag{9.7}$$

In particular, the generating function of the c sequence of the production matrix of $[a_{cd}(\eta), \eta]$ for the case $k = 0$ is $-\tan(\eta)$.

9.3 $\left[\frac{1}{\sqrt{\tilde{\Lambda}(1+k^2)}}\mathbf{dc}\left(\frac{\eta}{\sqrt{1+k^2}},k\right),\eta\right] \equiv [a_{\mathbf{dc}}(\eta),\eta]$

We consider the case when the solution takes the form of the sequence $a_{\mathrm{dc}}(\eta)$

$$a_{\mathrm{dc}}(\eta) = \frac{1}{\sqrt{(k^2+1)\Lambda}} + \frac{\eta^2(1-k)}{2(k^2+1)\sqrt{(k^2+1)\Lambda}} + \frac{\eta^4\left(k^2-6k+5\right)}{24\left(k^2+1\right)^2\sqrt{(k^2+1)\Lambda}} + \frac{\eta^6\left(-k^3+47k^2-107k+61\right)}{720\left(k^2+1\right)^3\sqrt{(k^2+1)\Lambda}} + \frac{\eta^8\left(k^4-412k^3+2142k^2-3116k+1385\right)}{40320\left(k^2+1\right)^4\sqrt{(k^2+1)\Lambda}} + O\left(\eta^9\right). \tag{9.8}$$

We can construct the proper exponential Riordan array $\left[\frac{1}{\sqrt{\tilde{\Lambda}(1+k^2)}}\mathrm{dc}\left(\frac{\eta}{\sqrt{1+k^2}},k\right),\eta\right] \equiv [a_{\mathrm{dc}}(\eta),\eta]$ having coefficient matrix given by

$$A = \begin{pmatrix} \frac{1}{\sqrt{(k^2+1)\Lambda}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{(k^2+1)\Lambda}} & 0 & 0 & 0 & 0 \\ \frac{(1-k)\Lambda}{((k^2+1)\Lambda)^{3/2}} & 0 & \frac{1}{\sqrt{(k^2+1)\Lambda}} & 0 & 0 & 0 \\ 0 & \frac{(3-3k)\Lambda}{((k^2+1)\Lambda)^{3/2}} & 0 & \frac{1}{\sqrt{(k^2+1)\Lambda}} & 0 & 0 \\ \frac{(k-5)(k-1)}{(k^2+1)^2\sqrt{(k^2+1)\Lambda}} & 0 & \frac{(6-6k)\Lambda}{((k^2+1)\Lambda)^{3/2}} & 0 & \frac{1}{\sqrt{(k^2+1)\Lambda}} & 0 \\ 0 & \frac{5(k-5)(k-1)}{(k^2+1)^2\sqrt{(k^2+1)\Lambda}} & 0 & -\frac{10(k-1)\Lambda}{((k^2+1)\Lambda)^{3/2}} & 0 & \frac{1}{\sqrt{(k^2+1)\Lambda}} \end{pmatrix}.$$

The production matrix of A in terms of k where $0<k<1$ which corresponds to non elementary function is given by

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1-k}{k^2+1} & 0 & 1 & 0 & 0 \\ 0 & \frac{2-2k}{k^2+1} & 0 & 1 & 0 \\ -\frac{2\left(k^2-1\right)}{\left(k^2+1\right)^2} & 0 & \frac{3-3k}{k^2+1} & 0 & 1 \\ 0 & -\frac{8\left(k^2-1\right)}{\left(k^2+1\right)^2} & 0 & \frac{4-4k}{k^2+1} & 0 \\ . \end{pmatrix}.$$

The production matrix of B in terms of $k=-1,0,1$ are given by

$$C = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 1 \\ 0 & 8 & 0 & 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

The generating function of the c sequence of the production matrix of $[a_{\mathrm{dc}}(\eta),\eta]$

in terms of modulus k is given by

$$\frac{(k-1)\mathrm{nc}\left(\frac{\eta}{\sqrt{k^2+1}}\middle| k\right)\mathrm{sc}\left(\frac{\eta}{\sqrt{k^2+1}}\middle| k\right)}{\sqrt{k^2+1}\mathrm{dc}\left(\frac{\eta}{\sqrt{k^2+1}}\middle| k\right)}. \tag{9.9}$$

In particular, the generating function of the c sequence of the production matrix of $[a_{\mathrm{dc}}(\eta), \eta]$ for the case $k=0$ is $-\tan(\eta)$.

Chapter 10

Riordan arrays and the analytical solution of the Quantum-Mechanical Oscillator Equation

10.1 Overview

This chapter will begin with an introduction to the classical mechanical oscillator followed by the quantum mechanical oscillator. Some key aspects of the harmonic oscillator in both classical and quantum mechanics will be highlighted. The relationship between the analytic solution of the quantum mechanical oscillator and the Riordan arrays associated to the Hermite polynomials will be illustrated. The chapter will culminate with verifying the solution of the quantum mechanical oscillator from the basis of the theory of Riordan arrays using the symbolic computational algebra system *Mathematica.*

10.2 The Harmonic Oscillator

The simple harmonic oscillator in classical mechanics corresponds to the case of the harmonic oscillator for which there is assumed to be no friction. Harmonic motion constitutes one of the most important examples of motion in physics.

Some examples of mechanical systems that can illustrate the phenomena of the harmonic oscillator include pendulums, acoustical systems and the spring-mass dashpot. Motion under a harmonic potential which in simple cases is often a mass attached to a spring is determined starting from the solution to Newton's equation given by

$$F = ma = m\frac{d^2x}{dt^2} = -\frac{dV(x)}{dx} = -kx \tag{10.1}$$

where k is a force constant for the spring connecting the masses, and $V(x) = \frac{1}{2}kx^2$ is the harmonic potential also known as **(Hooke's Law)**. In general vibrations having a restoring force equivalent to the Hooke's law arises from a simple harmonic oscillator. The potential for the harmonic oscillator occurs as the natural solution satisfied by every potential having small oscillations at the minimum. The importance of the harmonic oscillator can be highlighted by the fact that a very significant number of potentials that occur in nature have small potentials at the minimum which also includes systems relevant to quantum mechanics. The harmonic motion is a prerequisite to understanding more rigorous applications. The general solution of equation (10.1) is given by

$$x = x_M \cos(\omega t - \phi).$$

The displacement of the particle from the equillibrium position is sinusoidally varying with amplitude x_M and angular frequency ω. The connection between ω, the mass and force constant is given by

$$\omega = \sqrt{\frac{k}{m}}.$$

The kinetic energy of the mass is

$$T = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{p^2}{2m}$$

where $p = m\frac{dx}{dt}$ is the momentum of the particle.

Since $k = m\omega^2$, the total energy is given by

$$
\begin{aligned}
E &= T + V(x) \\
&= \frac{p^2}{2m} + \frac{1}{2}kx^2 \\
&= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.
\end{aligned}
$$

The formula for the total energy of the classical harmonic oscillator will later be used to write the Shrödinger equation for the quantum harmonic oscillator.

Furthermore, the harmonic oscillator for the case of a spring mass dashpot satisfies the differential equation:

$$m\ddot{y}(t) + ky(t) = 0. \tag{10.2}$$

The equation (10.2) can be rewritten as

$$\ddot{y}(t) + \omega_0^2 y(t) = 0\ ,\ \omega_0 = \sqrt{\frac{k}{m}}. \tag{10.3}$$

The solution of (10.3) can be expressed in the form $y(t) = e^{rt}$ for which there exists a solution if $r = \pm i\omega_0$ [90].

It can equally be established that the motion arising from a simple harmonic oscillator is sinusoidal about the equilibrium point, with a constant amplititude and a constant frequency based on the alternative form of its solution given by

$$y(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \tag{10.4}$$

10.3 Quantum-Mechanical Oscillator Equation and Riordan arrays

The quantum harmonic oscillator is characterized by the Shrödinger equation [96, 97]. The Shrödinger equation for a harmonic oscillator may be derived from the classical spring-mass potential in section (10.2) to give

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\Psi(x) = E\Psi(x)$$

where

$$\Psi(x) = Ce^{-\alpha x^2/2}$$

such that $\Psi \to 0$ as $x \to \infty$. By substituting the function Ψ into the Shrödinger equation and applying the boundary conditions, it follows that the ground state energy is given by

$$E_0 = \frac{\hbar\omega}{2}$$

where $\hbar$ is the Planck's constant.

The classical forces in a chemical bond can also be described in terms of a spring-like approximation or Hooke's law type forces. The simplest atom can be described by the PDE

$$-ihu_t = \frac{h^2}{2m}\triangle u + \frac{e^2}{r}u \tag{10.5}$$

where the potential $\frac{e^2}{r}$ is a variable coefficient. The free Shrödinger equation is given by

$$-i\frac{\delta u}{\delta t} = \frac{1}{2}\triangle u \tag{10.6}$$

in three dimensions where $h = m = 1$ and the potential term $\frac{e^2}{r}$ dropped in (10.5) . The **quantization** arises from the boundary conditions when solving the Shrödinger equation. Molecules can be viewed dynamically through the lens of both vibrational and rotational dynamics, which can be formulated and determined from a quantum mechanical framework. The **quantum-mechanical oscillator** which is sometimes referred to as the quantum harmonic oscillator represents the quantum mechanical description of the classical harmonic oscillator. At very high energy chemical bonds or forces holding spring-like systems reach their dissociation limits resulting in a significant deviation from Hooke's law and it therefore becomes important to consider the quantum mechanics of a harmonic oscillator. A peculiarity of defining an oscilator quantum mechanically is the probability of determining the position of a particle based on the solution of the Shrödinger's equation which leads to the occcurence of eigen values. Another key aspect of the quantum-mechanical oscillator is that it is one of the few quantum-mechanical systems for which there exists analytic solution to the model system that describes it. A real chemical bond normally exists in three dimensions, but it is still possible to get useful results from the one-dimensional case. The quantum mechanical oscillator in one dimension is given by

$$-iu_t = u_{xx} - x^2u \qquad (-\infty < x < \infty) \tag{10.7}$$

The condition

$$u \to 0 \text{ as } x \to \pm\infty \tag{10.8}$$

since at $x = \pm\infty$ the potential energy becomes infinite. In particular, the condition (10.8) is required to derive the eigen functions [109]. Furthermore, seperating the variables in

$$u = T(t)v(x)$$

results to

$$-i\frac{T'}{T} = \frac{v''-x^2v}{v} = -\lambda$$

where λ is considered the energy of the harmonic oscillator with $v(x)$ satisfying the ODE

$$v'' + (\lambda - x^2)v = 0. \tag{10.9}$$

The simplest solution of (10.9) is $e^{-x^2/2}$ [109]. That is, for the case $\lambda = 1$ we have that $v(x) = e^{-x^2/2}$ and for any other λ the general solution to (10.9) is

$$v(x) = w(x)e^{-x^2/2}. \tag{10.10}$$

In (10.10) we can construct the proper exponential Riordan array

$$\left[e^{-x^2/2}, x\right]$$

which is the coefficient matrix of the Hermite polynomials He_n defined and constructed in section (6.1.2.1) .

Using the FTRA we can determine that

$$\left[e^{-x^2/2}, x\right] w(x) \equiv e^{x^2/2}w(x) = v(x).$$

By substituting (10.10) into (10.9) results to the Hermite differential equation in terms of w given by

$$w'' - 2xw' + (\lambda - 1)w = 0 \tag{10.11}$$

The solution of the PDE (10.7) derived in [109] is given explicitly by

$$u_k(x,t) = e^{-i(2k+1)t} H_k(x) e^{-x^2/2}. \tag{10.12}$$

It can be rewritten as

$$u_k(x,t) = e^{-i\lambda t} H_k(x) e^{-x^2/2}$$

where only odd cases of λ are considered in (10.9) so that $\lambda = 2k+1$ with $k = 0, 1, 2, 3, ...$

The coefficient matrix arising from the Hermite polynomials $H_k(x)$ can be represented by the exponential Riordan array

$$H = \left[e^{-z^2}, 2z\right],$$

which has been defined and constructed in section (6.1.2.2).

The generating function of the polynomial sequence of H is given by

$$H_k(x) = \left[e^{-z^2}, 2z\right] \cdot e^{xz} = e^{-z^2} \cdot e^{x2z}.$$

The first five terms of the sequence $H_k(x)$ are given by

$$\left(1, 2x, 4x^2 - 2, 8x^3 - 12x, 16x^4 - 48x^2 + 12, ...\right).$$

The Riordan array H associated to the Hermite polynomials is embedded in the solution (10.12) to the quantum mechanical oscillator (10.2) such that

$$\begin{aligned} u(x,t) &= e^{-i(2k+1)t}e^{-z^2}\cdot e^{x2z}e^{-x^2/2} \\ &= e^{-z^2}\cdot e^{x2z}e^{-i(2k+1)t}e^{-x^2/2} \\ &= \left[e^{-z^2},2z\right]e^{xz}\cdot e^{-i(2k+1)t}e^{-x^2/2} \\ &= \left[e^{-z^2},2z\right]\cdot e^{xz-x^2/2-i(2k+1)t} \\ &= (u_k(x,t))_{k\in\mathbb{N}}. \end{aligned}$$

That is

$$(u_k(x,t))_{k\in\mathbb{N}} = \begin{pmatrix} 1 & & & & \\ 0 & 2 & & & \\ -2 & 0 & 4 & & \\ 0 & -12 & 0 & 8 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} e^{-it-\frac{x^2}{2}} \\ xe^{-3it-\frac{x^2}{2}} \\ x^2e^{-5it-\frac{x^2}{2}} \\ x^3e^{-7it-\frac{x^2}{2}} \\ \vdots \end{pmatrix}$$

$$= (u_0(x,t),u_1(x,t),u_2(x,t),u_3(x,t),...)^T.$$

The solution of the PDE associated to the Quantum-Mechanical oscillator can therefore be represented as the product of an infinite lower triangular Riordan matrix and an infinite dimensional vector derived from the sequence $x^ke^{-i(2k+1)t-\frac{x^2}{2}}$ indexed by k or $x^ke^{-i\lambda t-\frac{x^2}{2}}$ indexed by k with λ representing the energy levels at odd positions. Thus, there exists a Riordan matrix representation for the solution of the PDE associated to the Quantum-Mechanical Oscillator.

Furthermore, using the terms of the Hermite polynomial sequence we can expand the function $u_k(x,t)$ to get

$$\left(e^{-it}+e^{-3it}z+e^{-5it}z^2+e^{-7it}z^3+e^{-9it}z^4+O\left(z^5\right)\right)(e^{-\frac{x^2}{2}}+2e^{-\frac{x^2}{2}}xz+ e^{-\frac{x^2}{2}}\left(4x^2-2\right)z^2+e^{-\frac{x^2}{2}}\left(8x^3-12x\right)z^3+e^{-\frac{x^2}{2}}\left(16x^4-48x^2+12\right)z^4+O\left(z^5\right)). \tag{10.13}$$

That is the product (10.13) using the rules of f.p.s becomes:

$$e^{-\frac{x^2}{2}-it} + 2xze^{-\frac{x^2}{2}-3it} + \left(4x^2-2\right)z^2e^{-\frac{x^2}{2}-5it} + \left(8x^3-12x\right)z^3e^{-\frac{x^2}{2}-7it} + \left(16x^4-48x^2+12\right)z^4e^{-\frac{x^2}{2}-9it} + O\left(z^5\right). \quad (10.14)$$

Equation (10.14) forms the sequence of solutions corresponding to (10.12) which is given by

$$\left(e^{-\frac{x^2}{2}-it}, 2xe^{-\frac{x^2}{2}-3it}, \left(4x^2-2\right)e^{-\frac{x^2}{2}-5it}, \left(8x^3-12x\right)e^{-\frac{x^2}{2}-7it}, \left(16x^4-48x^2+12\right)e^{-\frac{x^2}{2}-9it}, ...\right). \quad (10.15)$$

The sequence (10.15) with the generating function

$$e^{-i(2n+1)t-\frac{x^2}{2}+2xz-z^2}, n = 0, 1, 2, 3, ...$$

can be verified using *Mathematica* to represent the family of solutions of the PDE describing the motion of the quantum mechanical oscillator (10.7) . The solution (10.15) can also be written in terms of the Euler's identity to have the sequence

$$P_u = \left((\cos(t) + i\sin(t))\,e^{-1-\frac{x^2}{2}}, 2x\,(\cos(3t) + i\sin(3t))\,e^{-1-\frac{x^2}{2}}, ...\right).$$

The *Mathematica* symbolic code to derive the solution of the quantum mechical oscillator using Riordan array technique is given below.

Expand [Table [n!SeriesCoefficient [Exp [−z^2] Exp[x2z], {z, 0, n}] , {n, 0, 4}]]

$\left\{1, 2x, -2+4x^2, -12x+8x^3, 12-48x^2+16x^4\right\}$

Table[Exp[−I(2n + 1)t], {n, 0, 4}]

$\left\{e^{-it}, e^{-3it}, e^{-5it}, e^{-7it}, e^{-9it}\right\}$

SeriesData [z, 0, {e^(−it), e^(−3it), e^(−5it), e^(−7it), e^(−9it)} , 0, 5, 1]

$e^{-it} + e^{-3it}z + e^{-5it}z^2 + e^{-7it}z^3 + e^{-9it}z^4 + O[z]^5$

$\boldsymbol{\left\{1, 2x, -2+4x^2, -12x+8x^3, 12-48x^2+16x^4\right\} \mathrm{Exp}\left[\frac{-x^2}{2}\right]}$

$$\left\{e^{-\frac{x^2}{2}}, 2e^{-\frac{x^2}{2}}x, e^{-\frac{x^2}{2}}\left(-2+4x^2\right), e^{-\frac{x^2}{2}}\left(-12x+8x^3\right), e^{-\frac{x^2}{2}}\left(12-48x^2+16x^4\right)\right\}$$

SeriesData $\left[z, 0, \left\{e^{-\frac{x^2}{2}}, 2e^{-\frac{x^2}{2}}x, e^{-\frac{x^2}{2}}\left(-2+4x^2\right), e^{-\frac{x^2}{2}}\left(-12x+8x^3\right), e^{-\frac{x^2}{2}}\left(12-48x^2+16x^4\right)\right\}, 0, 5, 1\right]$

$$e^{-\frac{x^2}{2}}+2e^{-\frac{x^2}{2}}xz+e^{-\frac{x^2}{2}}\left(-2+4x^2\right)z^2+e^{-\frac{x^2}{2}}\left(-12x+8x^3\right)z^3+e^{-\frac{x^2}{2}}\left(12-48x^2+16x^4\right)z^4+O[z]^5$$

$\left\{e^{-it}, e^{-3it}, e^{-5it}, e^{-7it}, e^{-9it}\right\} * \left\{e^{-\frac{x^2}{2}}, 2e^{-\frac{x^2}{2}}x, e^{-\frac{x^2}{2}}\left(-2+4x^2\right), e^{-\frac{x^2}{2}}\left(-12x+8x^3\right), e^{-\frac{x^2}{2}}\left(12-48x^2+16x^4\right)\right\}$

$$\left\{e^{-it-\frac{x^2}{2}}, 2e^{-3it-\frac{x^2}{2}}x, e^{-5it-\frac{x^2}{2}}\left(-2+4x^2\right), e^{-7it-\frac{x^2}{2}}\left(-12x+8x^3\right), e^{-9it-\frac{x^2}{2}}\left(12-48x^2+16x^4\right)\right\}$$

SeriesData $\left[z, 0, \left\{e^{-it-\frac{x^2}{2}}, 2e^{-3it-\frac{x^2}{2}}x, e^{-5it-\frac{x^2}{2}}\left(-2+4x^2\right), e^{-7it-\frac{x^2}{2}}\left(-12x+8x^3\right), e^{-9it-\frac{x^2}{2}}\left(12-48x^2+16x^4\right)\right\}, 0, 5, 1\right]$

$$e^{-it-\frac{x^2}{2}}+2e^{-3it-\frac{x^2}{2}}xz+e^{-5it-\frac{x^2}{2}}\left(-2+4x^2\right)z^2+e^{-7it-\frac{x^2}{2}}\left(-12x+8x^3\right)z^3+e^{-9it-\frac{x^2}{2}}\left(12-48x^2+16x^4\right)z^4+O[z]^5$$

$u:=\left\{e^{-it-\frac{x^2}{2}}, 2e^{-3it-\frac{x^2}{2}}x, e^{-5it-\frac{x^2}{2}}\left(-2+4x^2\right), e^{-7it-\frac{x^2}{2}}\left(-12x+8x^3\right), e^{-9it-\frac{x^2}{2}}\left(12-48x^2+16x^4\right)\right\}$

Table $\left[\text{Simplify}\left[-ID[u[[n]],t]==D[u[[n]],\{x,2\}]-x^2u[[n]]\right],\{n,1,5\}\right]$

$\{\text{True}, \text{True}, \text{True}, \text{True}, \text{True}\}$

Simplify $\left[\text{Exp}[-I(2n+1)t]\text{Exp}\left[-z^2\right]\text{Exp}[x2z]\text{Exp}\left[\frac{-x^2}{2}\right]\right]$

$$e^{-i(1+2n)t-\frac{x^2}{2}+2xz-z^2}$$

$v:=\text{Table}\left[n!\text{SeriesCoefficient}\left[e^{-i(1+2n)t-\frac{x^2}{2}+2xz-z^2},\{z,0,n\}\right],\{n,0,4\}\right]$

Table $\left[\text{Simplify}\left[-ID[v[[n]],t]==D[v[[n]],\{x,2\}]-x^2v[[n]]\right],\{n,1,5\}\right]$

$\{\text{True}, \text{True}, \text{True}, \text{True}, \text{True}\}$

Chapter 11

Filter Design and Riordan arrays

11.1 Introduction

Filter design constitutes an important area known as signal processing in electronics. A signal in electronics can simply be described as a physical quantity usually represented by its waveform that transmits information from one point to another. The operation on a signal in order to remove unwanted components is known as filtering. Filters are important in both analog and digital electronics. Filters can be designed in a variety of ways in mathematics using some classical polynomial sequences. Some key areas in which the application of filters can be found are in the seperation of signal components with different frequency ranges such as in the telephone, in the suppression of disturbances such as narrow band interference or noise of the signal spectrum such as equalizer, sound controller. The concepts of a filter's systems functions, the order of a filter, types of filters, time and group delay are discussed below.

11.1.1 Systems Function of a Filter

The **System functions** (also known as transfer functions or network function) which are uniquely realizable by an electrical network implementing electrotechnology describes the input versus the output behavior of a given system using mathematically defined functions based on the frequency domain. Equivalently a transfer function can be defined as the ratio of the Laplace transforms of its output and input signals. The voltage transfer function $H(s)$ of a filter can therefore be written as:

$$H(s) = \frac{V_{OUT}(s)}{V_{IN}(s)}. \tag{11.1}$$

In particular having knowledge about the transfer function's magnitude or gain at each frequency is a prerequisite to understanding how well the filter can deal with variations arising from signals transmitted at various frequency values. The filter's effect on the magnitude and phase of the signal is given by

$$|H(j\omega)| = \left|\frac{V_{OUT}(j\omega)}{V_{IN}(j\omega)}\right|. \tag{11.2}$$

On the other hand the phase is given by

$$arg H(j\omega) = arg \frac{V_{OUT}(j\omega)}{V_{IN}(j\omega)} \tag{11.3}$$

where $V_{IN}(s)$ and $V_{OUT}(s)$ are the input and output signal voltages respectively, s is the complex frequency variable, $j\omega$, where j is equal to $\sqrt{-1}$, such that ω is the radian frequency $(2\pi f)$.

$$F(s) = \frac{U_2(s)}{U_1(s)}. \tag{11.4}$$

In terms of its power series expansion this can be described as

$$F(s) = \frac{a_0 + a_1 \cdot s + ... + a_m \cdot s^m}{b_0 + b_1.s... + b_n.s^n} \tag{11.5}$$

with $m:$ $number\ of\ zeros, n:$ $number\ of\ poles, n \geq m, s = \sigma + j\omega:$ $complex\ frequency$ if $\sigma = 0:$ $s \rightarrow j\omega \Rightarrow F(s) \rightarrow F(j\omega)$.

11.1.2 The Order of a Filter

The **order** of a filter is the highest power of the variable s in its transfer function. Calculations with respect to the order of a filter are important for several reasons arising from the fact that it is directly related to the number of components in the filter, and therefore has an effect on its cost, its physical size, and the complexity of the design task. It is on this basis that higher-order filters are more expensive, take up more space, and are associated with complexities in their design structures. The primary advantage of a higher-order filter is that it will have a steeper roll-off slope than a similar lower-order filter. Before actually calculating the amplitude response of the network, we can see that at very low frequencies (small values of s), the numerator becomes very small, as do the first two terms of the denominator. Thus, as s approaches zero, the numerator approaches zero, the denominator approaches one, and $H(s)$ approaches zero. Similarly, as the input frequency approaches infinity, $H(s)$ also becomes progressively smaller, because the denominator increases with the square of frequency while the numerator increases linearly with frequency. Therefore, $H(s)$ will have its maximum value at some frequency between zero and infinity, and will decrease at frequencies above and below the peak.

11.1.3 Types of Filters

Filters are named based on the frequency range that are permitted for transmission through them and in the process effectively seeking to prevent any unwanted frequencies in the system. Based on this principle there are three categories of filters which are generally known:

- The Low Pass Filters are designed to ensure that only low frequencies are transmitted having a range from OHz to the cut-off frequency inclusive. The cut-off frequency is simply the frequency at which transmission occurs.
- The High Pass Filter are designed to ensure that only high frequency signals from its cut-off frequency point and higher up to infinity are transmitted thereby discarding any lower frequencies.
- The Band Pass Filter are designed to allow signals falling within a defined range consisting of a minimum and maximum point are allowed transmission. On the other hand, a **stopband** is the range of frequencies which fall between a specified upper and lower limit, through which a circuit such as a filter does not allow signals to pass.

There are different kinds of filters comprising the Butterworth, the Chebyshev Filters, Elliptic Filters, Bessel Filters. In this chapter we shall focus on the Bessel Filters and the Elliptic Filters. The elliptic filters are constructed from elliptic functions which have been treated in previous chapters. The Bessel Filters are constructed using the reverse Bessel polynomials defined and constructed in section (6.1.3).

11.1.4 Time and Group Delay

In signal processing, there is a time period allowed for the waves to propagate through the system. For a time delay λ, the signal will be delayed without any altering of its amplitude. The delay results in a phase lag ϕ which is directly proportional to its frequency ω. Time delay is the preferable measure of the filter's time effects rather than the phase delay relative to the frequency domain. A phase delay $\phi\{f\}$(radians) at a frequency f has a time delay given by

$$\triangle t = \frac{\phi\{f\}}{2\pi f}.$$

The filter tends to delay the signals in its passband by uniform amounts when considering the time delay. A filter whose phase is a linear function of frequency has constant time delay with applications found in the circuits of control and communication systems [108]. The Bessel filter produces an excellent approximation to a constant time delay.

The **group delay** $D(\omega)$ for a given smooth phase function describes the time delay of its amplitude envelope of a sinusoid at the frequency ω. The bandwidth of the amplitude envelope for the group delay under consideration is restricted to a frequency interval over which the phase response is approximately linear [82].

11.2 Elliptic Filter

Elliptic filters are constructed using the approximating function known as the **elliptic rational function**, usually denoted by $R_n(\xi, x)$. In the context of signal processing the elliptic rational function $R_n(\xi, x)$ is most widely used as the most suitable minimax approximation of a unit square pulse. Chebyshev polynomials can be used for the approximation of the unit square pulse [68] but with a less accurate approximation compared to the elliptic rational functions of the same order.

The rational function $R_n(\xi, x)$ can be implemented using the elliptic Jacobi function *cd* [68]. The key properties of the elliptic rational function which is satisfied by the elliptic Jacobi function are as follows:

- The equiripple property and $\mid R_n(\xi, x) \mid \leq 1$ whenever $\mid x \mid \leq 1$

- The largest value of $\min(\mid R_n(\xi,x) \mid\leq 1)$ for $\mid x \mid\geq \xi > 1$

- The minimal order n.

The elliptic rational function [58] is defined by

$$R_n(k,x) = \text{cd}\left(\frac{u}{M},\lambda\right) \qquad x = \text{cd}(u,k) \tag{11.6}$$

The explicit definition in Lutovac [68] is given by

$$R_n(\xi,x) = \text{cd}\left(n\frac{K\left(\frac{1}{L_n(\xi)}\right)}{K\left(\frac{1}{\xi}\right)}\text{cd}^{-1}\left(x,\frac{1}{\xi}\right),\frac{1}{L_n(\xi)}\right) \tag{11.7}$$

Another alternative definition of the elliptic rational function [68] is given in parametric form

$$R_n(\xi,x) = \text{cd}\left(nwK\left(\frac{1}{L_n(\xi)}\right),\frac{1}{L_n(\xi)}\right) \tag{11.8}$$

where K is the complete elliptic integral of the first kind, n is the order, ξ is the selectivity factor($\xi > 1$), w is an intermediate variable. The notation $L_n(\xi)$ represents the discrimination factor which is the minimal value of $\mid R_n(\xi,x) \mid$ for $\mid x \geq \xi \mid$.

The connection between the elliptic filter and Riordan arrays is based on examining the Jacobi Riordan array

$$E_{R_n} = \left[\text{cd}(\frac{u}{M},\lambda),u\right].$$

The coefficient matrix of E_{R_n} is given by

$$E_c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\lambda-1}{M^2} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{3(\lambda-1)}{M^2} & 0 & 1 & 0 & 0 & 0 \\ \frac{5\lambda^2-6\lambda+1}{M^4} & 0 & \frac{6(\lambda-1)}{M^2} & 0 & 1 & 0 & 0 \\ 0 & \frac{5(5\lambda^2-6\lambda+1)}{M^4} & 0 & \frac{10(\lambda-1)}{M^2} & 0 & 1 & 0 \\ \frac{61\lambda^3-107\lambda^2+47\lambda-1}{M^6} & 0 & \frac{15(5\lambda^2-6\lambda+1)}{M^4} & 0 & \frac{15(\lambda-1)}{M^2} & 0 & 1 \end{pmatrix}.$$

The corresponding production matrix of E_c is given by

$$E_p = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\lambda-1}{M^2} & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{2(\lambda-1)}{M^2} & 0 & 1 & 0 & 0 \\ \frac{2\left(\lambda^2-1\right)}{M^4} & 0 & \frac{3(\lambda-1)}{M^2} & 0 & 1 & 0 \\ 0 & \frac{8\left(\lambda^2-1\right)}{M^4} & 0 & \frac{4(\lambda-1)}{M^2} & 0 & 1 \\ \frac{16(\lambda-1)((\lambda-1)\lambda+1)}{M^6} & 0 & \frac{20\left(\lambda^2-1\right)}{M^4} & 0 & \frac{5(\lambda-1)}{M^2} & 0 \end{pmatrix}.$$

The c generating function of E_p corresponding to E_c is given by

$$c(u, \lambda, M) = \frac{(\lambda-1)\operatorname{nd}\left(\frac{u}{M} \middle| \lambda\right)\operatorname{sd}\left(\frac{u}{M} \middle| \lambda\right)}{M\operatorname{cd}\left(\frac{u}{M} \middle| \lambda\right)}.$$

Since as the r generating function is 1, therefore the c generating function that determines the first column E_{R_n} determines the Jacobi Riordan array E_c. Note that u, λ, M are defined based on (11.6,11.7,11.8).

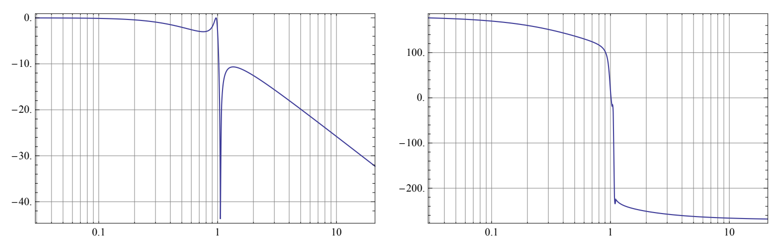


Figure 11.1: Bode plot of a third order lowpass elliptic filter model with cutoff frequency $\omega_c = 1$

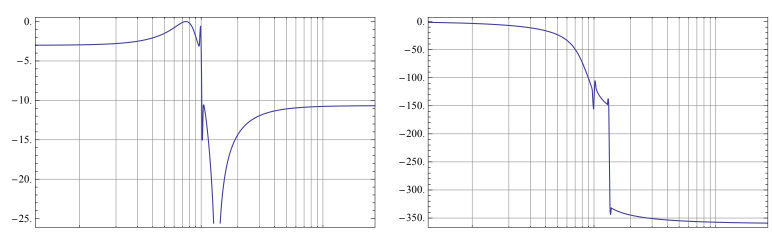


Figure 11.2: Bode plot of a fourth order lowpass elliptic filter model with cutoff frequency $\omega_c = 1$

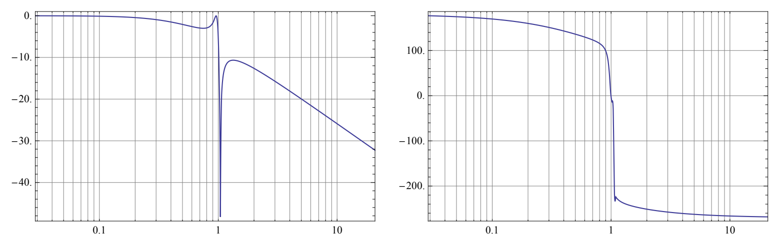


Figure 11.3: Bode Plot of a fifth order lowpass elliptic filter model with cutoff frequency $\omega_c = 1$

11.3 Bessel Filter

11.3.1 Introduction

The origin of the Bessel filter is based on the mathematical theory of Bessel functions put forward in the 1824 memoir of the German mathematician and astronomer Friedrich Bessel$(1784 - 1846)$ [116]. In signal processing, the Bessel filter is characterized by a maximally flat group and phase delay such that output response results to an unchanged wave shape of the filtered signals in the passband [111, 7].

In particular a low-pass Bessel filter is characterised by its transfer function given by

$$H(s) = \frac{\theta_n(0)}{\theta_n(s/\omega_c)} \tag{11.9}$$

where $n = 1, 2, 3, ...$ and ω_c is the cut-off frequency and $\theta_n(s)$ is the reverse Bessel polynomial . We consider the case when the cut-off frequency $\omega_c = 1$ so that the transfer function of the Bessel filter (11.9) becomes

$$H(s) = \frac{\theta_n(0)}{\theta_n(s)}. \tag{11.10}$$

Using the techniques from Riordan array we determine the general term of the transfer function (11.10) of order n.

The Bessel polynomial used for High pass filters has the denominator polynomial given by

$$\{\frac{1}{3} + s + s^2, \frac{1}{15} + \frac{2}{5}s + s^2 + s^3, \frac{1}{105} + \frac{2}{21}s + \frac{9}{21}s^2 + s^3 + s^4, ...\}$$

On the other hand, the reverse Bessel polynnomial used for Low pass denominator polynomial given by

$$\{1+s+\frac{1}{3}s^3, 1+s+\frac{2}{5}s^2+\frac{1}{15}s^3, 1+s+\frac{9}{21}s^2+\frac{2}{21}s^3+\frac{1}{105}s^4, ...\}.$$

11.3.2 Bessel filter using $\left[\frac{1}{\sqrt{1-2t}}, 1-\sqrt{1-2t}\right]$

The coefficient matrix corresponding to the Bessel polynomials can be represented by the exponential Riordan array belonging to the derivative subgroup given by

$$\left[\frac{1}{\sqrt{1-2t}}, 1-\sqrt{1-2t}\right],$$

where

$$\frac{d}{dt}(1-\sqrt{1-2t}) = \frac{1}{\sqrt{1-2t}}.$$

In terms of Riordan arrays, the reverse Bessel polynomials corresponds to the polynomial sequence determined by

$$\left[\frac{1}{\sqrt{1-2t}}, 1-\sqrt{1-2t}\right] \cdot e^{st}, \tag{11.11}$$

which is equivalent to the Riordan array bivariate generating function such that

$$\frac{1}{\sqrt{1-2t}} e^{s(1-\sqrt{1-2t})}. \tag{11.12}$$

In particular using (11.12) , let $\frac{1}{\sqrt{1-2t}} e^{s(1-\sqrt{1-2t})} = \theta(s,t)$ such that

$$\theta(s,t) = \sum_{n=0}^{\infty} \theta_n(s) t^n. \tag{11.13}$$

Also, using equation (11.12) when $s = 0$ and let $\frac{1}{\sqrt{1-2t}} = \theta(t)$ such that

$$\theta(t) = \sum_{n=0}^{\infty} \theta_n(0) t^n. \tag{11.14}$$

We have that from (11.13) & (11.14)

$$\frac{\theta(t)}{\theta(s,t)} = \frac{\sum_{n=0}^{\infty} \theta_n(0) t^n}{\sum_{n=0}^{\infty} \theta_n(s) t^n}. \tag{11.15}$$

The right hand side of (11.13) for a given Bessel filter of order n can be rewritten as

$$\frac{\theta_0(0)+\theta_1(0)+\theta_2(0)+\theta_3(0)+...+\theta_n(0)}{\theta_0(s)+\theta_1(s)+\theta_2(s)+\theta_3(s)+...+\theta_n(s)} = \frac{\theta_0(0)}{\theta_0(s)}+\frac{\theta_1(0)}{\theta_1(s)}+\frac{\theta_2(0)}{\theta_2(s)}+\frac{\theta_3(0)}{\theta_3(s)}+...+\frac{\theta_n(0)}{\theta_n(s)}. \tag{11.16}$$

Since as a formal power series can be interchanged with its associated sequence we have that (11.16) corresponds to the sequence

$$\left(\frac{\theta_0(0)}{\theta_0(s)}, \frac{\theta_1(0)}{\theta_1(s)}, \frac{\theta_2(0)}{\theta_2(s)}, \frac{\theta_3(0)}{\theta_3(s)}, ..., \frac{\theta_n(0)}{\theta_n(s)}\right) = \left(\frac{\theta_n(0)}{\theta_n(s)}\right)_{n\in\mathbb{N}}. \tag{11.17}$$

The sequence in (11.17) corresponds to a sequence of transfer functions of the form given in (11.9) for $n = 1, 2, 3, ...$ where n is the order of the Bessel filter.

Therefore, starting from the exponential Riordan array

$$R = \left[\frac{1}{\sqrt{1-2t}}, 1-\sqrt{1-2t}\right]$$

we can obtain a sequence of transfer function of the Bessel filter. The r and c generating functions of R are

$$r(z) = \frac{1}{4z-2} + \frac{1}{\sqrt{1-2z}} \qquad c(z) = \frac{1}{(z-1)^2}.$$

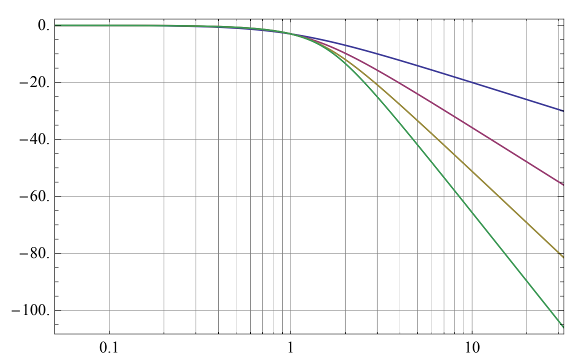


Figure 11.4: The plot shows the gain of low pass Bessel Filter of order $n = 1, 2, 3, 4$

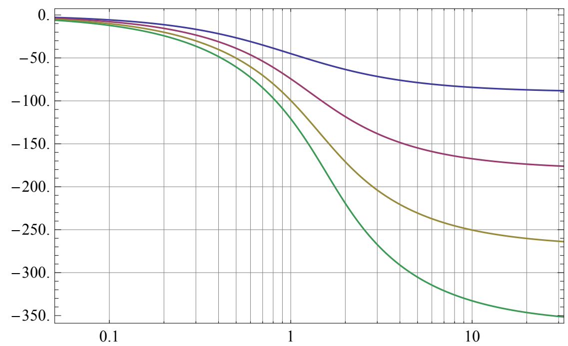


Figure 11.5: The plot shows the group delay of low pass Bessel filter of order $n = 1, 2, 3, 4$

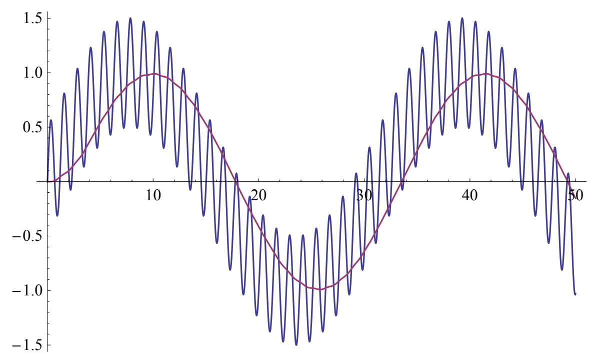


Figure 11.6: The plot illustrates the effect of low pass Bessel filter of order 4 that filters out high frequency noise from a sinusoidal signal

11.3.3 Bessel filter using $\left[\frac{1}{\sqrt{1-4t}}, 1-\sqrt{1-4t}\right]$

The coefficient array of $R = \left[\frac{1}{\sqrt{1-4t}}, 1-\sqrt{1-4t}\right]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 12 & 12 & 4 & 0 & 0 & 0 \\ 120 & 120 & 48 & 8 & 0 & 0 \\ 1680 & 1680 & 720 & 160 & 16 & 0 \\ 30240 & 30240 & 13440 & 3360 & 480 & 32 \end{pmatrix}.$$

The production matrix of A is given by

$$B = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 \\ 4 & 4 & 2 & 0 & 0 \\ 12 & 12 & 6 & 2 & 0 \\ 48 & 48 & 24 & 8 & 2 \\ 240 & 240 & 120 & 40 & 10 \end{pmatrix}.$$

The r and c generating functions of R are given by

$$r(z) = \frac{2}{1-z} \qquad c(z) = \frac{2}{(1-z)^2}.$$

The polynomial sequence from A is given by

$$1, 2s+2, 4s^2+12s+12, 8s^3+48s^2+120s+120, 16s^4+160s^3+720s^2+1680s+1680,$$
$$32s^5 + +480s^4 + 3360s^3 + 13440s^2 + 30240 + 30240, ... \quad (11.18)$$

By using the elements staring from the second row of the first column of the matrix A as the numerator and the elements of the polynomial sequence in (11.18) starting from the second element, we get the transfer function of order $1, 2, 3, 4$ respectively in the form

$$C = \frac{2}{2s+2}, \frac{12}{4s^2+12s+12}, \frac{120}{8s^3+48s^2+120s+120}, \frac{1680}{16s^4+160s^3+720s^2+1680s+1680}.$$

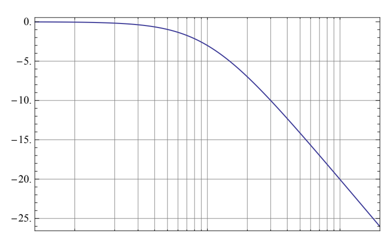


Figure 11.7: The plot shows the gain of low pass Bessel Filter of order $n = 1, 2, 3, 4$ using $\left[\frac{1}{\sqrt{1-4t}}, 1-\sqrt{1-4t}\right]$

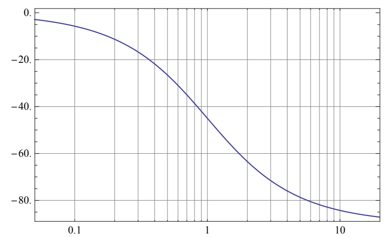


Figure 11.8: The plot shows the group delay of low pass Bessel Filter of order $n = 1, 2, 3, 4$ using $\left[\frac{1}{\sqrt{1-4t}}, 1 - \sqrt{1-4t}\right]$

11.3.4 Bessel filter using $\left[\frac{1}{\sqrt{1-6t}}, 1 - \sqrt{1-6t}\right]$

The coefficient array of $R = \left[\frac{1}{\sqrt{1-6t}}, 1 - \sqrt{1-6t}\right]$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 0 \\ 27 & 27 & 9 & 0 & 0 & 0 \\ 405 & 405 & 162 & 27 & 0 & 0 \\ 8505 & 8505 & 3645 & 810 & 81 & 0 \\ 229635 & 229635 & 102060 & 25515 & 3645 & 243 \end{pmatrix}.$$

The production matrix of A is given by

$$B = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 \\ 6 & 6 & 3 & 0 & 0 \\ 18 & 18 & 9 & 3 & 0 \\ 72 & 72 & 36 & 12 & 3 \\ 360 & 360 & 180 & 60 & 15 \end{pmatrix}.$$

The r and c generating functions of R are given by

$$r(z) = \frac{3}{1-z} \qquad c(z) = \frac{3}{(1-z)^2}.$$

The polynomial sequence from A is given by

$$1, 3s{+}3, 9s^2{+}27s{+}27, 27s^3{+}162s^2{+}405s{+}405, 81s^4{+}810s^3{+}3645s^2{+}8505s{+}8505, 243s^5{+} \\ {+}3645s^4 + 25515s^3 + 102060s^2 + 229635s + 229635, ... \quad (11.19)$$

By using the elements staring from the second row of the first column of the matrix A as the numerator and the elements of the polynomial sequence in (11.19) starting from the second element, we get the transfer function of order

$1, 2, 3, 4$ respectively in the form

$$C = \frac{3}{3s+3}, \frac{27}{9s^2+27s+27}, \frac{405}{27s^3+162s^2+405s+405}, \frac{8505}{81s^4+810s^3+3645s^2+8505s+8505}$$

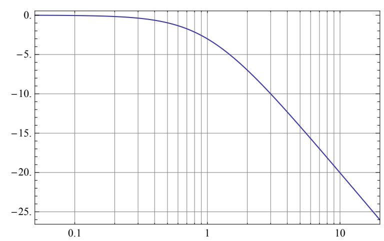


Figure 11.9: The plot shows the gain of low pass Bessel Filter of order $n = 1, 2, 3, 4$ using $\left[\frac{1}{\sqrt{1-6t}}, 1-\sqrt{1-6t}\right]$

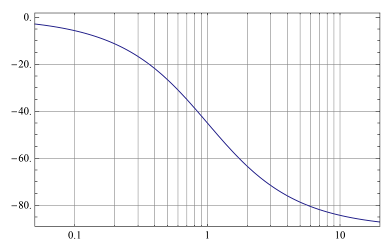


Figure 11.10: The plot shows the group delay of low pass Bessel Filter of order $n = 1, 2, 3, 4$ using $\left[\frac{1}{\sqrt{1-6t}}, 1-\sqrt{1-6t}\right]$

Similar results can be verified for the case of $\left[\frac{1}{\sqrt{1-8t}}, 1-\sqrt{1-8t}\right]$ and $\left[\frac{1}{\sqrt{1-10t}}, 1-\sqrt{1-10t}\right]$

11.3.5 Bessel filter using $\left[\frac{1}{\sqrt[3]{1-3t}}, 1-\sqrt[3]{1-3t}\right]$

The coefficient array of $R = \left[\frac{1}{\sqrt[3]{1-3t}}, 1-\sqrt[3]{1-3t}\right]$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 4 & 4 & 1 & 0 & 0 & 0 \\ 28 & 28 & 9 & 1 & 0 & 0 \\ 280 & 280 & 100 & 16 & 1 & 0 \\ 3640 & 3640 & 1380 & 260 & 25 & 1 \end{pmatrix}.$$

The production matrix of A is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 \\ 12 & 12 & 5 & 1 & 0 \\ 60 & 60 & 27 & 7 & 1 \\ 360 & 360 & 168 & 48 & 9 \end{pmatrix}.$$

The r and c generating functions of R are given by

$$r(z) = \frac{1}{(1-z)^2} \qquad c(z) = \frac{1}{(1-z)^3}.$$

The polynomial sequence from A is given by

$$1, s+1, s^2+4s+4, s^3+9s^2+28s+28, s^4+16s^3+100s^2+280s+280, s^5+25s^4+260s^3 + 1380s^2 + 3640s + 3640, ... \quad (11.20)$$

By using the elements staring from the second row of the first column of the matrix A as the numerator and the elements of the polynomial sequence in (11.20) starting from the second element, we get the transfer function of order $1, 2, 3, 4$ respectively in the form

$$C = \frac{1}{1s+1}, \frac{4}{s^2+4s+4}, \frac{28}{s^3+9s^2+28s+28}, \frac{280}{s^4+16s^3+100s^2+280s+280}$$

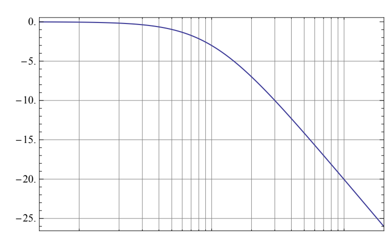


Figure 11.11: The plot shows the gain of low pass Bessel Filter of order $n = 1, 2, 3, 4$ using $\left[\frac{1}{\sqrt[3]{1-3t}}, 1 - \sqrt[3]{1-3t}\right]$

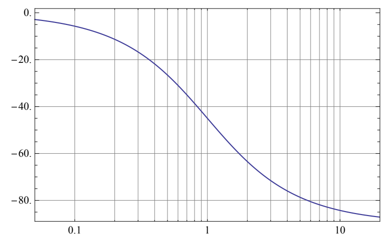


Figure 11.12: The plot shows the group delay of low pass Bessel Filter of order $n = 1, 2, 3, 4$ using $\left[\frac{1}{\sqrt[3]{1-3t}}, 1 - \sqrt[3]{1-3t}\right]$

11.3.6 Bessel filter using $\left[\frac{1}{\sqrt[4]{1-4t}}, 1 - \sqrt[4]{1-4t}\right]$

The coefficient array of $R = \left[\frac{1}{\sqrt[4]{1-4t}}, 1 - \sqrt[4]{1-4t}\right]$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 5 & 5 & 1 & 0 & 0 & 0 \\ 45 & 45 & 12 & 1 & 0 & 0 \\ 585 & 585 & 177 & 22 & 1 & 0 \\ 9945 & 9945 & 3240 & 485 & 35 & 1 \end{pmatrix}.$$

The production matrix of A is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 4 & 4 & 1 & 0 & 0 \\ 20 & 20 & 7 & 1 & 0 \\ 120 & 120 & 48 & 10 & 1 \\ 840 & 840 & 360 & 88 & 13 \end{pmatrix}.$$

The r and c generating functions of R are given by

$$r(z) = \frac{1}{(1-z)^3} \qquad c(z) = \frac{1}{(1-z)^4}.$$

The polynomial sequence from A is given by

$$1, s+1, s^2+5s+5, s^3+12s^2+45s+45, s^4+22s^3+177s^2+585s+585, s^5+35s^4+485s^3+3240s^2 + 9945s + 9945, \ldots \quad (11.21)$$

By using the elements staring from the second row of the first column of the matrix A as the numerator and the elements of the polynomial sequence in (11.21) starting from the second element, we get the transfer function of order

1, 2, 3, 4 respectively in the form

$$C = \frac{1}{1s+1}, \frac{5}{s^2+5s+5}, \frac{45}{s^3+12s^2+45s+45}, \frac{585}{s^4+22s^3+177s^2+585s+585}$$

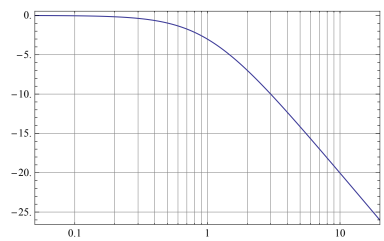


Figure 11.13: The plot shows the gain of low pass Bessel Filter of order $n = 1, 2, 3, 4$ using $\left[\frac{1}{\sqrt[4]{1-4t}}, 1 - \sqrt[4]{1-4t}\right]$

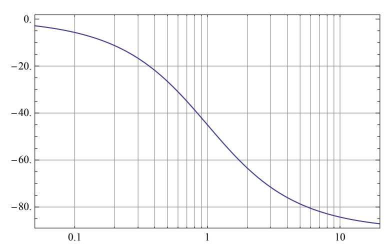


Figure 11.14: The plot shows the group delay of low pass Bessel Filter of order $n = 1, 2, 3, 4$ using $\left[\frac{1}{\sqrt[4]{1-4t}}, 1 - \sqrt[4]{1-4t}\right]$

11.4 Passive Systems

11.4.1 Introduction

Passive systems are most often associated to control systems engineering and circuit network theory in analog electronics, represent components used in the implementation of such systems which consumes energy but do not produce energy. The design and modeling of filters for passive systems corresponding to single-input single-output linesr time-invariant systems are achieved using certain types polynomials. In particular, such systems are implemented by having their transfer function as the ratio between two successive recursively defined polynomials characterized by certain properties. These polynomials include the

Fibonacci, Morgan-Voyce, Lucas and Jacobsthal type. **Loss-less systems** and **Positive real relaxation systems** constitute particular classes of passive systems designed from these polynomials. A loss-less one port circuit conserves the total energy from the generating source flowing through its elements. Loss-less systems can be derived based on the ratio between successive polynomials of the Fibonacci or Lucas polynomials . On the other hand, relaxation systems are passive systems that are designed using only subsets of possible passive components comprising resistor, capacitors and inductors. Relaxation systems are designed from the ratio of successive Morgan-Voyce or Jacobsthal polynomials. Given that f_n, l_n, b_n and B_n are the Fibonacci,Lucas polynomials, Morgan-Voyce polynomials of the first and second kind respectively of degree n, the following theorems in [42] below describe the way these polynomials achieve loss-less systems.

Theorem 11.4.1 $G(s) = \frac{f_n(s)}{f_{n+1}(s)}$ *represents the transfer function of a controllable and observable loss-less system of order* n.

Theorem 11.4.2 $G(s) = \frac{f_n(s)}{l_n(s)}$ *represents the transfer function of a controllable and observable loss-less system of order* n.

Theorem 11.4.3 $G_n(s) = \frac{b_n(s)}{b_{n+1}(s)}$ *represents the transfer function of a controllable and observable positive real and relaxation system of order* $n+1$.

Theorem 11.4.4 $G_n(s) = \frac{B_n(s)}{B_{n+1}(s)}$ *represents the transfer function of a controllable and observable positive real and relaxation system of order* $n+1$.

We redefine the concepts of passive systems represented by the four theorems above in terms of Riordan arrays associated to the Fibonacci and the Morgan-Voyce polynomials.

11.4.2 Riordan arrays from the Fibonacci and Morgan-Voyce polynomials

The Fibonacci polynomials has the coefficient matrix represented by the ordinary Riordan array given by

$$f = \left(\frac{1}{1-t^2}, \frac{t}{1-t^2}\right).$$

The coefficient matrix of f is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 1 & 0 \\ 1 & 0 & 6 & 0 & 5 & 0 & 1 \end{pmatrix}.$$

The production matrix of f is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 2 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

Using the Riordan array f we show that it represents the family of Fibonacci polynomials as follows:

$$\begin{aligned}
[t^n]\left(\frac{1}{1-t^2},\frac{t}{1-t^2}\right)\cdot\frac{1}{1-ts} &= [t^n]\left(\frac{1}{1-t^2}\cdot\frac{1}{1-s\frac{t}{1-t^2}}\right)\\
&= [t^n]\frac{1}{1-t^2}\sum_{k=0}^{\infty}\left(\frac{st}{1-t^2}\right)^k\\
&= [t^n]\frac{1}{1-t^2}\sum_{k=0}^{\infty}\frac{s^k t^k}{1-t^2}\\
&= [t^n]\sum_{k=0}^{\infty}s^k\frac{t^k}{(1-t^2)^{k+1}}\\
&= [t^n]\sum_{k=0}^{\infty}s^k t^k(1-t^2)^{-(k+1)}\\
&= [t^{n-k}]\sum_k\sum_j\binom{k+1+j-1}{j}(-1)^j(-t^2)^j s^k\\
&= [t^{n-k}]\sum_k\sum_j\binom{k+j}{j}t^{2j}s^k\\
&= \sum_{k=0}^{n}\binom{k+\frac{n-k}{2}}{\frac{n-k}{2}}s^k\cdot\frac{(1+(-1)^{n-k})}{2}\\
&= \sum_{k=0}^{n}\binom{\frac{n+k}{2}}{k}s^k\cdot\frac{(1+(-1)^{n-k})}{2}\\
&= (f_n(s))_{n\in\mathbb{N}}.
\end{aligned}$$

Remark: Note $2j = n-k \implies j=\frac{n-k}{2}$ and

$$\binom{n}{k}=\binom{n}{n-k} \implies \frac{n+k}{2}-\frac{n-k}{2}=k.$$

Similarly, the Morgan-Voyce polynomials B_n and b_n have been defined and constructed in section (6.1.5) .

11.4.3 The Transfer function $G(s)$ of Passive Sytems and Riordan arrays

The transfer functions $G(s)$ for the controllable and observable loss-less systems and relaxation systems in (11.4.1) can be rewritten in terms of Riordan arrays below.

- The transfer function in (11.4.1) corresponds to

$$G(s) = \frac{f_n(s)}{f_{n+1}(s)} \equiv \frac{[t^n] \sum_{k=0}^{\infty} \frac{1}{1-t^2} \left(\frac{t}{1-t^2}\right)^k s^k}{[t^{n+1}] \sum_{k=0}^{\infty} \frac{1}{1-t^2} \left(\frac{t}{1-t^2}\right)^k s^k}$$

where

$$f_n(s) = \sum_{k=0}^{\lfloor \frac{(n+1)}{2} \rfloor} \binom{n-k-1}{k} s^{n-2k-1}.$$

- The transfer function in (11.4.2) corresponds to

$$G(s) = \frac{f_n(s)}{l_n(s)} \equiv \frac{[t^n] \sum_{k=0}^{\infty} \frac{1}{1-t^2} \left(\frac{t}{1-t^2}\right)^k s^k}{l_n s^k}.$$

where the denominator $l_n(s)$ equals

$$2^{-n} \left[\left(s - \sqrt{s^2+4}\right)^n + \left(s + \sqrt{s^2+4}\right)^n \right].$$

Furthermore, the Lucas polynomials are related to the triangular array **A027960** given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 5 & 0 & 1 & 0 & 0 \\ 2 & 0 & 9 & 0 & 6 & 0 & 1 & 0 \\ 0 & 7 & 0 & 14 & 0 & 7 & 0 & 1 \end{pmatrix}.$$

The array L has the production matrix given by

$$Lp = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 12 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

- The transfer function in (11.4.3) corresponds to

$$G_n(s) = \frac{b_n(s)}{b_{n+1}(s)} \equiv \frac{[t^n]\sum_{k=0}^{\infty} \frac{1}{1-t}\left(\frac{t}{(1-t)^2}\right)^k s^k}{[t^{n+1}]\sum_{k=0}^{\infty} \frac{1}{1-t}\left(\frac{t}{(1-t)^2}\right)^k s^k}$$

where the numerator and the denominator are respectively given by

$$b_n(s) = \sum_{k=0}^{n} \binom{n+k}{n-k} s^k \;\&\; b_{n+1}(s) = \sum_{k=0}^{n+1} \binom{n+k+1}{n-k+1} s^k.$$

- The transfer function in (11.4.4) corresponds to

$$G_n(s) = \frac{B_n(s)}{B_{n+1}(s)} \equiv \frac{[t^n]\sum_{k=0}^{\infty} \frac{1}{(1-t)^2}\left(\frac{t}{(1-t)^2}\right)^k s^k}{[t^{n+1}]\sum_{k=0}^{\infty} \frac{1}{(1-t)^2}\left(\frac{t}{(1-t)^2}\right)^k s^k}$$

where the numerator and the denominator are respectively given by

$$B_n(s) = \sum_{k=0}^{n} \binom{n+k+1}{n-k} s^k \;\&\; B_{n+1}(s) = \sum_{k=0}^{n+1} \binom{n+k+2}{n-k+1} s^k.$$

11.5 Conclusion

- In this chapter some examples of families of polynomial sequences that implement the transfer function used for Bessel filters and which have not been investigated previously in the domain of signal processing have been constructed using Riordan array techniques.

- **Conjecture:** Riordan arrays of the form

$$\left[\frac{1}{\sqrt[m]{1-mt}}, 1-\sqrt[m]{1-mt}\right]\cdot e^{st} \ \& \ \left[\frac{1}{\sqrt{1-2nt}}, 1-\sqrt{1-2nt}\right]\cdot e^{st}$$

  where $m, n \in \mathbb{Z}$ such that $m \geq 2$ and $n \geq 1$. provide the mechanics for generating various families of polynomials that define the transfer function used for implementing the Bessel filter.

- The transfer function of some Passive systems has been redefined in terms of the generic element of the Riordan arrays of the Fibonacci and the Morgan-Voyce polynomials.

Chapter 12

Conclusions and Future Directions

The main contributions of this thesis are :

- This thesis is the first work looking at elliptic functions and Riordan arrays. A principal discovery is that exponential Riordan arrays and elliptic functions are very well suited for each other, via the reversion process which can be applied to functions defined by integrals, which is a feature of both elliptic functions and exponential Riordan arrays. In particular, for the case of exponential Riordan arrays the A-sequence(also known as the c sequence) is also very important to establishing its relationship to elliptic functions.

- In the study of Riordan arrays constructed from elliptic functions, some interesting parameterized elliptic Riordan arrays have emerged. These are interesting because of the polynomial nature of the entries, and in other cases because of the shape of the resulting production matrices. Further work can still be done on such elliptic Riordan arrays. But one key aspect is that this is the first work to present parameterized polynomial entries of Riordan arrays.

- Another contribution of this work is the identification and classification of some Riordan arrays which constitute the solutions of systems of differential equations (Sturm-Liouville equations).

- As it is indicative of the thesis title several areas of applications of elliptic Riordan arrays and other Riordan arrays have been investigated from Chapter 7 to Chapter 11 of this thesis. These areas include the KdV, non linear electrical transmission lines, cosmology, elliptic filters arising from elliptic Riordan arrays on one hand and the Bessel filters and the solution to the quantum-mechanical oscillator arising from some of the non-elliptic Riordan arrays that are solutions of the Sturm-Liouville equations. In particular, the identification of the role of Riordan arrays in filter design constitutes a novel application that can further be developed at a later stage.

Furthermore, in this thesis, we have established connections between elliptic functions and Riordan arrays. Several examples of elliptic Riordan arrays have been presented in chapters 2, 3, 4. In Chapter 2, some of the Jacobi Riordan arrays were associated to important combinatorial and algebraic interpretations for the cases of the nodulus $m = \{0, 1\}$. In addition, some previously unknown new sequences and triangular arrays having interesting algebraic structures corresponding to the associated trigonometric and hyperbolic forms of the elliptic Jacobi functions were found. These new sequences and arrays are currently in the process of been added to the OEIS. Further investigation will seek to extend the possible combinatorial significance arising from Jacobi Riordan arrays for the general case $0 \leq m \leq 1$.

In future research work the columns of elliptic Riordan arrays and their production matrices will be examined for those with polynomial sequences in m having first term 1. The hankel matrices of such polynomial sequences will then be computed to determine whether they are non-negative which is a key criterion for the moment sequences derived from orthogonal polynomial sequences. The original polynomials can then be determined from the formula (1.12)

$$P_n(z) = \frac{\triangle_n(z)}{\triangle_{n-1}}.$$

The inverse of the coefficient matrix of $P_n(z)$ and its tri-diagonal matrix can then be examined. For elliptic dn function in $[\mathrm{dn}, \mathrm{sn}]$ we have the following initial results listed below. The elliptic dn function can be expanded to get

$$\left\{1, 0, 1, 0, 5 - 4m, 0, 16m^2 - 76m + 61, 0, -64m^3 + 1104m^2 - 2424m + 1385 + ...\right\}.$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4m-5 & 0 & 1 & 0 & 0 & 0 \\ 9 & 0 & 4m-14 & 0 & 1 & 0 & 0 \\ 0 & 64m^2-144m+89 & 0 & 20m-30 & 0 & 1 & 0 \\ -225 & 0 & 64m^2-244m+439 & 0 & 20m-55 & 0 & 1 \end{pmatrix}.$$

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5-4m & 0 & 1 & 0 & 0 & 0 \\ 5-4m & 0 & 14-4m & 0 & 1 & 0 & 0 \\ 0 & 16m^2-76m+61 & 0 & 30-20m & 0 & 1 & 0 \\ 16m^2-76m+61 & 0 & 16m^2-256m+331 & 0 & 55-20m & 0 & 1 \end{pmatrix}.$$

$$Pdmat = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 4-4m & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 & 1 \\ 0 & 0 & 0 & 16-16m & 0 \end{pmatrix}$$

where Pdmat represents the production matrix of P^{-1}.

A particular focus of this thesis has been to identify the applications of Riordan arrays and elliptic functions. The examples provided in this work for elliptic Riordan arrays and other non-elliptic Riordan arrays associated to the solutions of some Sturm-Liouville differential equations have provided new insights to potential areas of applications. In this regard, this work has examined the application of elliptic functions with their corresponding elliptic Riordan arrays to the solution of the non-linear KdV wave equations in Chapter 7. This was followed by applications of Riordan arrays to the PDE modelling the wave-like phenomena of the quantum-mechanical harmonic oscillator in quantum physics and Low pass electrical transmission in Chapter 10 and Chapters 8 respectively. These application areas involved establishing the Riordan arrays linked to the exact solutions of the differential equations modeling these wave-like phenomena.

In Chapter 6, we presented examples of Riordan arrays as solutions to some Sturm-Liouville differential equations. On the other hand, consider the Lamé equation [64] which is a linear ordinary second-order differential equation in the complex domain given by

$$\frac{d^2w}{dz^2} = [A + B\wp(z)]\,w,$$

where $\wp$ is the Weierstrass function and A and B are constants. The corresponding Jacobi form of the Lamé is given by

$$\frac{d^2w}{du^2} = \left[C + D\mathrm{sn}^2 u\right] w.$$

The next step of investigation will seek to determine appropriate elliptic Riordan arrays that are solutions to the Weierstrass and Jacobi forms of the Lamé equation.

In filter design, the reverse Bessel polynomials in previous research has been presented as the only explicit family of polynomials used in the design of the Bessel filter [7]. By using the mechanics of Riordan arrays we determine new families of polynomials that can be used to construct the Bessel filter described in section (11.3). These families of polynomials were used to construct the transfer functions that produced the characteristic maximally flat linear phase response similar to that of the transfer function derived from the original reverse Bessel polynomials.

We note the role of the Jacobi elliptic function *cd* in determining the solution of the FRLW cosmological model in Chapter 9 and the elliptic filter in section (11.2). By constructing Appell type Riordan arrays using the *cd* elliptic function we find that the recurrence formula arising from the Riordan array can provide an alternative representation of the solution of the FRLW cosmological model and the elliptic filter.

The researcher Zhedhanov [125] has previously studied the relationship between elliptic functions and the Toda chain. On the other hand Barry [10] has established the relationship between Toda chain and exponential Riordan arrays by considering examples of Riordan arrays belonging to the Sheffer class of orthogonal polynomials. The natural question that arises whether we can use the knowledge from this thesis on elliptic Riordan arrays to investigate the possible connections between the Toda chain and elliptic Riordan arrays. Zhedanov starts his investigation of the elliptic Toda polynomials by determining the general formula for the recurrence coefficients such that

$$u_n(t) = w^2 n^2 (\wp(w(t+\beta)) - \wp(nw(t+\beta)+q))$$

and

$$b_n(t) = \mu_1 + w(n+1)\zeta(w(n+1)(t+\beta)+q) - wn\zeta(wn(t+\beta)+q) - \\ - (2n+1)w\zeta(w(t+\beta))$$

where w, β, q, μ_1 are arbitrary complex parameters. In addition, it is assumed that ω_1, ω_3 are arbitrary independent periods corresponding to the arbitrary parameters e_1, e_2, e_3 such that $e_1+e_2+e_3=0$. In particular, $u_n(t), b_n(t)$ satisfy the restricted Toda chain equations given by

$$\dot{u}_n = u_n(b_n - b_{n-1}),\ \dot{b}_n = u_{n+1} - u_n$$

with the condition that $u_0 = 0$ and the dot $(\cdot)$ in the equation represents differentiation w.r.t time t. An interesting result arises when all the roots coincide s.t $e_1 = e_2 = e_3 = 0$. It then follows that the elliptic functions degenerates to simple rational ones given by $\wp = \frac{1}{z^2}, \zeta(z) = \frac{1}{z}, \sigma(z) = z$. For the additional assumption that $w = 1, \beta = 0$

$$c_0(t) = \frac{t+q}{qt} = \frac{1}{t} + \frac{1}{q}$$

which results in the first term $\frac{1}{t}$ in $c_0(t)$ above generating the Laguerre polynomials such that

$$b_n(t) = -\frac{2n+1}{t},\ u_n(t) = \frac{n^2}{t^2},$$

where the recurrence coefficients corresponding to the Laguerre polynomials $L_n^{(0)}(-xt)$ is a solution of the restricted Toda chain for the initial conditions $c_o^{(0)}(t) = \frac{1}{t}$. In section (6.1.1), the connection between the Laguerre polynomials and Riordan arrays has been established. This initial review shows that the trigonometric and hyperbolic functions for appropriately chosen parameters arising from the degenerate cases of the elliptic recurrence coefficients presented in [125], can inspire further scope of studying the basis to determine if there exists some forms of the elliptic Riordan arrays that may satisfy the Toda chain equation.

Appendices

Appendix A

Appendix-Symbolic Code

1. How to do the reversion of the function $\frac{t}{1-t}$

$\textbf{Solve}\left[\frac{u}{1-u} == t, u\right]$

$\left\{\left\{u \to \frac{t}{1+t}\right\}\right\}$

$\textbf{fbar[t\_]:=}\frac{t}{1+t}$

$g\textbf{[t\_]:=}\frac{1}{1-t}$

$\textbf{Simplify[Composition[}g\textbf{, fbar][}t\textbf{]]}$

$1 + t$

2. Determining the Riordan matrix and row sums of

$$\left[\frac{d}{dz}sn^{-1}(z,m), sn^{-1}(z,m)\right]$$

$\textbf{Clear[}m\textbf{]}$

$A\textbf{:=Table}\left[\frac{n!}{k!}\textbf{SeriesCoefficient}\left[(D[\textbf{InverseJacobiSN}[z,m],z])(\textbf{InverseJacobiSN}[z,m])^k,\{z,0,n\}\right],\right.$

$\{n,0,6\},\{k,0,6\}]$

$\textbf{Table[}A\textbf{.Table[1, }\{n,0,6\}\textbf{]}, \{m,-1,1\}\textbf{]}$

$\textbf{MatrixForm[Simplify[}A\textbf{[[1;;5, 1;;5]]]]}$

$\{\{1,1,1,1,13,73,253\},\{1,1,2,5,20,85,520\},\{1,1,3,9,45,225,1575\}\}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1+m & 0 & 1 & 0 & 0 \\ 0 & 4(1+m) & 0 & 1 & 0 \\ 9+6m+9m^2 & 0 & 10(1+m) & 0 & 1 \end{pmatrix}$$

3. Determining the Riordan matrix and production matrix for

$$\left[\frac{d}{dz}isc(z,m), isc(z,m)\right].$$

```
Clear[m]
A:=Table[n!/k! SeriesCoefficient[(D[I JacobiSN[z,m]/JacobiCN[z,m], z]) (I JacobiSN[z,m]/JacobiCN[z,m])^k, {z,0,n}], {n,0,6}, {k,0,6}]
MatrixForm[A[[1;;5, 1;;5]]]
MatrixForm[Simplify[A[[1;;5, 1;;5]]]]
Simplify[MatrixForm[Inverse[A][[1;;4, 1;;4]].A[[2;;5, 1;;4]]]]
```

$$\begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -i(-2+m) & 0 & -i & 0 & 0 \\ 0 & 4(-2+m) & 0 & 1 & 0 \\ i\left(16-16m+m^2\right) & 0 & 10i(-2+m) & 0 & i \end{pmatrix}$$

$$\begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -i(-2+m) & 0 & -i & 0 & 0 \\ 0 & 4(-2+m) & 0 & 1 & 0 \\ i\left(16-16m+m^2\right) & 0 & 10i(-2+m) & 0 & i \end{pmatrix}$$

$$\begin{pmatrix} 0 & i & 0 & 0 \\ i(-2+m) & 0 & i & 0 \\ 0 & 3i(-2+m) & 0 & i \\ -3im^2 & 0 & 6i(-2+m) & 0 \end{pmatrix}$$

4. Predicting the generating functions for the Riordan matrix corresponding to the Bessel polynomials using the matrix generated from its general formula

$$\sum_{k-0}^{n}\left(\frac{(2n-k)!}{2^{n-k}(k)!(n-k)!}\right)x^k.$$

```
MatrixForm[Transpose[Table[SeriesCoefficient[Table[Sum[((2n-k)!/(2^(n-k)(k)!(n-k)!)) x^k, {k,0,n}], {n,0,6}],
{x,0,n}], {n,0,6}]]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 15 & 15 & 6 & 1 & 0 & 0 & 0 \\ 105 & 105 & 45 & 10 & 1 & 0 & 0 \\ 945 & 945 & 420 & 105 & 15 & 1 & 0 \\ 10395 & 10395 & 4725 & 1260 & 210 & 21 & 1 \end{pmatrix}$$

$$B := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 15 & 15 & 6 & 1 & 0 & 0 & 0 \\ 105 & 105 & 45 & 10 & 1 & 0 & 0 \\ 945 & 945 & 420 & 105 & 15 & 1 & 0 \\ 10395 & 10395 & 4725 & 1260 & 210 & 21 & 1 \end{pmatrix}$$

A:=Table[1, {n, 0, 6}]

B.A

$\{1, 2, 7, 37, 266, 2431, 27007\}$

A001515 E.g.f: $\frac{e^{1-\sqrt{1-2x}}}{\sqrt{1-2x}}$

B[[All, 1]]

$\{1, 1, 3, 15, 105, 945, 10395\}$

A001147 E.g.f: $\frac{1}{\sqrt{1-2x}}$

Solve $\left[\frac{\mathrm{Exp}\left[1-\sqrt{1-2x}\right]}{\sqrt{1-2x}} == \frac{1}{\sqrt{1-2x}}\mathrm{Exp}[y], y, \mathrm{Reals}\right]$

$\left\{\left\{y \to \mathrm{ConditionalExpression}\left[1-\sqrt{1-2x}, x < \frac{1}{2}\right]\right\}\right\}$

BES:=Table $\left[\frac{n!}{k!}\mathrm{SeriesCoefficient}\left[\left(\frac{1}{\sqrt{1-2x}}\right)\left(1-\sqrt{1-2x}\right)^k, \{x, 0, n\}\right],\right.$

$\{n, 0, 6\}, \{k, 0, 6\}]$

MatrixForm[BES]

MatrixForm[Inverse[BES][[;;6, ;;6]].BES[[2;;, ;;6]]]

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 15 & 15 & 6 & 1 & 0 & 0 & 0 \\ 105 & 105 & 45 & 10 & 1 & 0 & 0 \\ 945 & 945 & 420 & 105 & 15 & 1 & 0 \\ 10395 & 10395 & 4725 & 1260 & 210 & 21 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 6 & 6 & 3 & 1 & 0 & 0 \\ 24 & 24 & 12 & 4 & 1 & 0 \\ 120 & 120 & 60 & 20 & 5 & 1 \\ 720 & 720 & 360 & 120 & 30 & 6 \end{pmatrix}$$

5. Evaluating the transfer functions for the Bessel filter of order $1, 2, 3, 4$.

Simplify $\left[\mathrm{Table}\left[n!\mathrm{SeriesCoefficient}\left[\mathrm{Evaluate}\left[\left(\frac{1}{\sqrt{1-2t}}\right)\mathrm{Exp}\left[s\left(1-\sqrt{1-2t}\right)\right]\right], \{t, 0, n\}\right], \{n, 0, 4\}\right]\right]$

$\{1, 1+s, 3+3s+s^2, 15+15s+6s^2+s^3, 105+105s+45s^2+10s^3+s^4\}$

Simplify $\left[\textbf{Table}\left[n!\textbf{SeriesCoefficient}\left[\frac{1}{\sqrt{1-2t}}, \{t,0,n\}\right], \{n,0,4\}\right]\right]$

$\{1, 1, 3, 15, 105\}$

Simplify $\left[\frac{\textbf{Table}\left[n!\textbf{SeriesCoefficient}\left[\frac{1}{\sqrt{1-2t}},\{t,0,n\}\right],\{n,0,4\}\right]}{\textbf{Table}\left[n!\textbf{SeriesCoefficient}\left[\textbf{Evaluate}\left[\left(\frac{1}{\sqrt{1-2t}}\right)\textbf{Exp}\left[s\left(1-\sqrt{1-2t}\right)\right]\right],\{t,0,n\}\right],\{n,0,4\}\right]}\right]$

$\left\{1, \frac{1}{1+s}, \frac{3}{3+3s+s^2}, \frac{15}{15+15s+6s^2+s^3}, \frac{105}{105+105s+45s^2+10s^3+s^4}\right\}$

6. Extracting the sub-matrix from the first column of $[dc(z,m), z]$

A**:=Table** $\left[\frac{n!}{k!}\textbf{SeriesCoefficient}\left[\left(\frac{\textbf{JacobiDN}[z,m]}{\textbf{JacobiCN}[z,m]}\right)(z\text{^}k), \{z,0,n\}\right], \{n,0,6\}, \{k,0,6\}\right]$

MatrixForm[Transpose[Table[SeriesCoefficient[A**[[All, 1]], {**m**, 0,** n**}], {**n**, 0, 6}]]]**

MatrixForm[Simplify[A**[[1;;6, 1;;6]]]]**

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1-m & 0 & 1 & 0 & 0 & 0 \\ 0 & 2-2m & 0 & 1 & 0 & 0 \\ 2-2m^2 & 0 & 3-3m & 0 & 1 & 0 \\ 0 & 8-8m^2 & 0 & 4-4m & 0 & 1 \\ -16(-1+m)(1+(-1+m)m) & 0 & -20\left(-1+m^2\right) & 0 & 5-5m & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 61 & -107 & 47 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1-m & 0 & 1 & 0 & 0 & 0 \\ 0 & 3-3m & 0 & 1 & 0 & 0 \\ 5-6m+m^2 & 0 & 6-6m & 0 & 1 & 0 \\ 0 & 5\left(5-6m+m^2\right) & 0 & -10(-1+m) & 0 & 1 \end{pmatrix}$$

$$B{:=}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 61 & -107 & 47 & -1 & 0 & 0 & 0 \end{pmatrix}$$

T**:=Table[**B**[[2**n **+ 1]], {**n**, 0, 3}]**

G**:=MatrixForm[**T**]**

G

$\boldsymbol{T}.\textbf{Table}\left[m^n,\{n,0,6\}\right]$

$\textbf{MatrixForm}\left[\textbf{Transpose}\left[\textbf{Table}\left[\frac{1}{(-1)^n}\textbf{SeriesCoefficient}\left[T.\textbf{Table}\left[m^n,\{n,0,6\}\right],\{m,0,n\}\right],\{n,0,3\}\right]\right]\right]$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 5 & -6 & 1 & 0 & 0 & 0 & 0 \\ 61 & -107 & 47 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\left\{1, 1-m, 5-6m+m^2, 61-107m+47m^2-m^3\right\}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & 6 & 1 & 0 \\ 61 & 107 & 47 & 1 \end{pmatrix}$$

7. Determining the Dixonian Riordan matrix and production matrix of

$$[sm(z)', sm(z)]\,.$$

```
cm[z_]:=(3WeierstrassPPrime[z, {0, 1/27}] + 1)/(3WeierstrassPPrime[z, {0, 1/27}] − 1)
sm[z_]:=(6WeierstrassP[z, {0, 1/27}])/(1 − 3WeierstrassPPrime[z, {0, 1/27}])
Series[sm[z], {z, 0, 12}]
A:=Table[(n!/(k!))SeriesCoefficient[(D[sm[z], z])((sm[z])^k), {z, 0, n}],
{n, 0, 8}, {k, 0, 8}]
MatrixForm[A]
FullSimplify[MatrixForm[Inverse[A][[1;;5, 1;;5]].A[[2;;6, 1;;5]]]]
```

$$z-\frac{z^4}{6}+\frac{2z^7}{63}-\frac{13z^{10}}{2268}+O[z]^{13}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -60 & 0 & 0 & 1 & 0 & 0 & 0 \\ 160 & 0 & 0 & -140 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1840 & 0 & 0 & -280 & 0 & 0 & 1 & 0 \\ 0 & 0 & 10800 & 0 & 0 & -504 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 \\ 0 & -16 & 0 & 0 & 1 \\ 0 & 0 & -40 & 0 & 0 \end{pmatrix}$$

8. Determining the soliton solutions of the KdV equation using the proposed

form of the solution given by

$$a + bcn^2(\xi, m).$$

$\boldsymbol{D\left[a + b\text{JacobiCN}[\xi, m]^2, \{\xi, 2\}\right]}$

$\boldsymbol{-c\left(a + b\text{JacobiCN}[\xi, m]^2\right) - 3\left(a + b\text{JacobiCN}[\xi, m]^2\right)^2 + b\left(-2\text{JacobiCN}[\xi, m]^2\text{JacobiDN}[\xi, m]^2 +\right.}$

$\boldsymbol{\left.2m\text{JacobiCN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2 + 2\text{JacobiDN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2\right)}$

$b\left(-2\text{JacobiCN}[\xi, m]^2\text{JacobiDN}[\xi, m]^2 +\right.$

$\left.2m\text{JacobiCN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2 + 2\text{JacobiDN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2\right)$

$-c\left(a + b\text{JacobiCN}[\xi, m]^2\right) - 3\left(a + b\text{JacobiCN}[\xi, m]^2\right)^2 + b\left(-2\text{JacobiCN}[\xi, m]^2\text{JacobiDN}[\xi, m]^2 +\right.$

$\left.2m\text{JacobiCN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2 + 2\text{JacobiDN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2\right)$

$\boldsymbol{\text{FullSimplify}\left[-c\left(a + b\text{JacobiCN}[\xi, m]^2\right) - 3\left(a + b\text{JacobiCN}[\xi, m]^2\right)^2 + b\left(-2\text{JacobiCN}[\xi, m]^2 \cdot\right.\right.}$

$\boldsymbol{\left.\left.\text{JacobiDN}[\xi, m]^2 + 2m\text{JacobiCN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2 + 2\text{JacobiDN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2\right)\right]}$

$-a(3a + c) - b\text{JacobiCN}[\xi, m]^2\left(-2 + 6a + c + 3b\text{JacobiCN}[\xi, m]^2 + 4\text{JacobiDN}[\xi, m]^2\right) +$

$2b\text{JacobiDN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2$

$\boldsymbol{\text{Expand}\left[-a(3a + c) - b\text{JacobiCN}[\xi, m]^2\left(-2 + 6a + c + 3b\text{JacobiCN}[\xi, m]^2 + 4\text{JacobiDN}[\xi, m]^2\right)\right.}$

$\boldsymbol{\left.+2b\text{JacobiDN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2\right]}$

$-3a^2 - ac + 2b\text{JacobiCN}[\xi, m]^2 - 6ab\text{JacobiCN}[\xi, m]^2 - bc\text{JacobiCN}[\xi, m]^2 -$

$3b^2\text{JacobiCN}[\xi, m]^4 - 4b\text{JacobiCN}[\xi, m]^2\text{JacobiDN}[\xi, m]^2 + 2b\text{JacobiDN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2$

$\boldsymbol{-3a^2 - ac + 2b\text{JacobiCN}[\xi, m]^2 - 6ab\text{JacobiCN}[\xi, m]^2 - bc\text{JacobiCN}[\xi, m]^2 - 3b^2\text{JacobiCN}[\xi, m]^4 -}$

$\boldsymbol{4b\text{JacobiCN}[\xi, m]^2\text{JacobiDN}[\xi, m]^2 + 2b\text{JacobiDN}[\xi, m]^2\text{JacobiSN}[\xi, m]^2}$

$\boldsymbol{//.\left\{\text{JacobiSN}[\xi, m]^2 \to 1 - \text{JacobiCN}[\xi, m]^2, \text{JacobiDN}[\xi, m]^2 \to 1 - m^2\left(1 - \text{JacobiCN}[\xi, m]^2\right)\right\}}$

$-3a^2 - ac + 2b\text{JacobiCN}[\xi, m]^2 - 6ab\text{JacobiCN}[\xi, m]^2 - bc\text{JacobiCN}[\xi, m]^2 - 3b^2\text{JacobiCN}[\xi, m]^4 - 4b\text{JacobiCN}[\xi, m]^2 \left(1 - m^2 \left(1 - \text{JacobiCN}[\xi, m]^2\right)\right) + 2b \left(1 - \text{JacobiCN}[\xi, m]^2\right) \cdot \left(1 - m^2 \left(1 - \text{JacobiCN}[\xi, m]^2\right)\right)$

Expand $\left[-3a^2 - ac + 2b\text{JacobiCN}[\xi, m]^2 - 6ab\text{JacobiCN}[\xi, m]^2 - bc\text{JacobiCN}[\xi, m]^2 - 3b^2\text{JacobiCN}[\xi, m]^4 - 4b\text{JacobiCN}[\xi, m]^2 \left(1 - m^2 \left(1 - \text{JacobiCN}[\xi, m]^2\right)\right) + 2b \left(1 - \text{JacobiCN}[\xi, m]^2\right) \left(1 - m^2 \left(1 - \text{JacobiCN}[\xi, m]^2\right)\right)\right]$

$-3a^2 + 2b - ac - 2bm^2 - 4b\text{JacobiCN}[\xi, m]^2 - 6ab\text{JacobiCN}[\xi, m]^2 - bc\text{JacobiCN}[\xi, m]^2 + 8bm^2\text{JacobiCN}[\xi, m]^2 - 3b^2\text{JacobiCN}[\xi, m]^4 - 6bm^2\text{JacobiCN}[\xi, m]^4$

$-3a^2 + 2b - ac - 2bm^2 - 4b\text{JacobiCN}[\xi, m]^2 - 6ab\text{JacobiCN}[\xi, m]^2 - bc\text{JacobiCN}[\xi, m]^2 + 8bm^2\text{JacobiCN}[\xi, m]^2 - 3b^2\text{JacobiCN}[\xi, m]^4 - 6bm^2\text{JacobiCN}[\xi, m]^4 //.\text{JacobiCN}[\xi, m] \to \text{cn}$

$-3a^2 + 2b - ac - 4b\text{cn}^2 - 6ab\text{cn}^2 - bc\text{cn}^2 - 3b^2\text{cn}^4 - 2bm^2 + 8b\text{cn}^2m^2 - 6b\text{cn}^4m^2$

Table $\left[\text{Coefficient}\left[-3a^2 + 2b - ac - 4b\text{cn}^2 - 6ab\text{cn}^2 - bc\text{cn}^2 - 3b^2\text{cn}^4 - 2bm^2 + 8b\text{cn}^2m^2 - 6b\text{cn}^4m^2, \text{cn}, n\right], \{n, 0, 4\}\right]$

$\left\{-3a^2 + 2b - ac - 2bm^2, 0, -4b - 6ab - bc + 8bm^2, 0, -3b^2 - 6bm^2\right\}$

Solve $\left[-3a^2 + 2b - ac - 2bm^2 == 0\&\& - 4b - 6ab - bc + 8bm^2 == 0\&\& - 3b\text{^}2 - 6bm\text{^}2 == 0, \{a, b, c\}, \text{Reals}\right]$

Cases considered from list of solutions above after evaluation:

$\left\{a \to \text{ConditionalExpression}\left[-\frac{1}{6}\sqrt{16 - 16m^2 + 16m^4} - \frac{1}{6}\sqrt{16 - 64m^2 + 64m^4}, -\frac{1}{\sqrt{2}} < m < \frac{1}{\sqrt{2}}\right],\right.$
$b \to \text{ConditionalExpression}\left[-2m^2, -\frac{1}{\sqrt{2}} < m < \frac{1}{\sqrt{2}}\right],$
$\left.c \to \text{ConditionalExpression}\left[\sqrt{16 - 16m^2 + 16m^4}, -\frac{1}{\sqrt{2}} < m < \frac{1}{\sqrt{2}}\right]\right\}$

$\left\{a \to \text{ConditionalExpression}\left[-\frac{1}{6}\sqrt{16 - 16m^2 + 16m^4} + \frac{1}{6}\sqrt{16 - 64m^2 + 64m^4}, m > \frac{1}{\sqrt{2}} \| m < -\frac{1}{\sqrt{2}}\right],\right.$
$b \to \text{ConditionalExpression}\left[-2m^2, m > \frac{1}{\sqrt{2}} \| m < -\frac{1}{\sqrt{2}}\right],$
$\left.c \to \text{ConditionalExpression}\left[\sqrt{16 - 16m^2 + 16m^4}, m > \frac{1}{\sqrt{2}} \| m < -\frac{1}{\sqrt{2}}\right]\right\}$
$\left\{\left\{a \to \text{ConditionalExpression}\left[\frac{1}{6}\sqrt{16 - 16m^2 + 16m^4} + \frac{1}{6}\sqrt{16 - 64m^2 + 64m^4}, m > \frac{1}{\sqrt{2}} \| m < -\frac{1}{\sqrt{2}}\right],\right.\right.$

$b \to \text{ConditionalExpression}\left[-2m^2, m > \frac{1}{\sqrt{2}} \| m < -\frac{1}{\sqrt{2}}\right],$
$c \to \text{ConditionalExpression}\left[-\sqrt{16-16m^2+16m^4}, m > \frac{1}{\sqrt{2}} \| m < -\frac{1}{\sqrt{2}}\right]\}\}$

Clear[c]

$a + b\text{JacobiCN}[\xi, m]^2//.\xi \to x - ct$

$a + b\text{JacobiCN}[ct - x, m]^2$

$a + b\text{JacobiCN}[ct - x, m]^2//.\{a \to -\frac{1}{6}\sqrt{16-16m^2+16m^4} - \frac{1}{6}\sqrt{16-64m^2+64m^4}, b \to -2m^2,$
$c \to \sqrt{16-16m^2+16m^4}\}$

$-\frac{1}{6}\sqrt{16-16m^2+16m^4} - \frac{1}{6}\sqrt{16-64m^2+64m^4} - 2m^2\text{JacobiCN}\left[\sqrt{16-16m^2+16m^4}t - x, m\right]^2$

$a + b\text{JacobiCN}[ct - x, m]^2//.\{a \to -\frac{1}{6}\sqrt{16-16m^2+16m^4} + \frac{1}{6}\sqrt{16-64m^2+64m^4}, b \to -2m^2,$
$c \to \sqrt{16-16m^2+16m^4}\}$

$-\frac{1}{6}\sqrt{16-16m^2+16m^4} + \frac{1}{6}\sqrt{16-64m^2+64m^4} - 2m^2\text{JacobiCN}\left[\sqrt{16-16m^2+16m^4}t - x, m\right]^2$

$a + b\text{JacobiCN}[ct - x, m]^2//.\{a \to \frac{1}{6}\sqrt{16-16m^2+16m^4} + \frac{1}{6}\sqrt{16-64m^2+64m^4}, b \to -2m^2,$
$c \to -\sqrt{16-16m^2+16m^4}\}$

$\frac{1}{6}\sqrt{16-16m^2+16m^4} + \frac{1}{6}\sqrt{16-64m^2+64m^4} - 2m^2\text{JacobiCN}\left[\sqrt{16-16m^2+16m^4}t + x, m\right]^2$

$m{:=}0$
$-\frac{1}{6}\sqrt{16-16m^2+16m^4} - \frac{1}{6}\sqrt{16-64m^2+64m^4} - 2m^2\text{JacobiCN}\left[\sqrt{16-16m^2+16m^4}t - x, m\right]^2$

$-\frac{4}{3}$

$m{:=}1$
$-\frac{1}{6}\sqrt{16-16m^2+16m^4} + \frac{1}{6}\sqrt{16-64m^2+64m^4} - 2m^2\text{JacobiCN}\left[\sqrt{16-16m^2+16m^4}t - x, m\right]^2$

$-2\text{Sech}[4t - x]^2$

usol[x_, t_] $= -2\text{Sech}[4t - x]^2$

$-2\text{Sech}[4t - x]^2$

$D[\text{usol}[x, t], t] == 6\text{usol}[x, t]D[\text{usol}[x, t], x] - D[\text{usol}[x, t], \{x, 3\}]//\text{Simplify}$

True

m:=1

$\frac{1}{6}\sqrt{16-16m^2+16m^4}+\frac{1}{6}\sqrt{16-64m^2+64m^4}-2m^2\text{JacobiCN}\left[\sqrt{16-16m^2+16m^4}t+x,m\right]^2$

$\frac{4}{3}-2\text{Sech}[4t+x]^2$

$\text{usol}[\text{x\_},\text{t\_}]=\frac{4}{3}-2\text{Sech}[4t+x]^2$

$\frac{4}{3}-2\text{Sech}[4t+x]^2$

$D[\text{usol}[x,t],t]$==$6\text{usol}[x,t]D[\text{usol}[x,t],x]-D[\text{usol}[x,t],\{x,3\}]$//Simplify

True

$\text{xmin}=-8;\text{xmax}=8;$

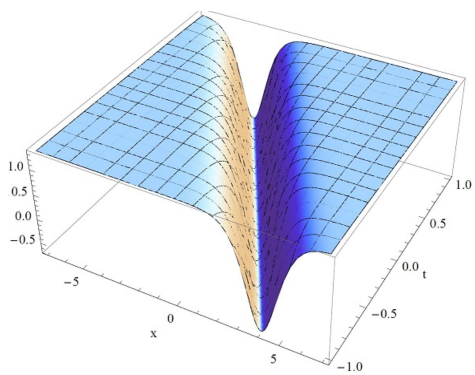
$\text{sol2}=\text{NDSolve}\left[\left\{D[u[x,t],t]==6u[x,t]D[u[x,t],x]-D[u[x,t],\{x,3\}],u[x,0]==\frac{4}{3}-2\text{Sech}[x]^2,\right.\right.$

$\left.\left.u[\text{xmin},t]==u[\text{xmax},t]\right\},u,\{x,\text{xmin},\text{xmax}\},\{t,-1,1\}\right]$

$\{\{u\to\text{InterpolatingFunction}[\{\{-8.,8.\},\{-1.,1.\}\},<>]\}\}$

$\text{Plot3D}[u[x,t]/.\text{Flatten}[\text{sol2}],\{x,-7,7\},\{t,-1,1\},\text{PlotPoints}\to 50,\text{PlotRange}\to\text{All},$

$\text{AxesLabel}\to\{\text{x},\text{t},\text{u}\}]$



A:=Table$\left[\frac{n!}{k!}\text{SeriesCoefficient}\left[\left(\text{JacobiCN}[\xi,m]^2\right)\left(\left(\text{Integrate}\left[\text{JacobiCN}[\xi,m]^2,\xi\right]\right)^k\right),\right.\right.$

$\left.\{\xi,0,n\}],\{n,0,5\},\{k,0,5\}\right]$

MatrixForm[Simplify[A]]

FullSimplify[MatrixForm[Inverse[A][[1;;5, 1;;5]].A[[2;;6, 1;;5]]]]

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -8 & 0 & 1 & 0 & 0 \\ 16 & 0 & -20 & 0 & 1 & 0 \\ 0 & 136 & 0 & -40 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & -6 & 0 & 1 & 0 \\ 0 & 0 & -12 & 0 & 1 \\ 0 & 0 & 0 & -20 & 0 \end{pmatrix}$$

Reduce[Tanh[u] == ξ, u, Reals]

$-1 < \xi < 1\&\&u ==$ ArcTanh[ξ]

rr[ξ_]:=Sech[ξ]^2

pp[ξ_]:=ArcTanh[ξ]

Composition[rr, pp][ξ]

$\frac{1}{\text{Composition[rr,pp][}\xi\text{]}}$

$1 - \xi^2$

$\frac{1}{1-\xi^2}$

9. Extracting the second column of the exponnential Riordan array $\left[cn(z,m)^2, sn(z,m)\right]$ to determine the submatrix from the second column corresponding to the expansion of $cn(z,m)^2 sn(z,m)$

h = Simplify [Table [n!SeriesCoefficient [JacobiCN[z,m]^{2}JacobiSN[z,m], {z, 0, n}] , {n, 0, 14}]]

Table[h[[2n]], {n, 1, 7}]

$\{0, 1, 0, -7 - m, 0, 61 + 74m + m^2, 0, -547 - 2739m - 681m^2 - m^3, 0, 4921 + 80788m + 85038m^2 + 6148m^3 + m^4, 0, -44287 - 2169797m - 6590134m^2 - 2324554m^3 - 55355m^4 - m^5, 0, 398581 + 55949982m + 413000631m^2 + 421686548m^3 + 60344691m^4 + 498222m^5 + m^6, 0\}$

$\{1, -7-m, 61+74m+m^2, -547-2739m-681m^2-m^3, 4921+80788m$
$+85038m^2+6148m^3+m^4, -44287-2169797m-6590134m^2-2324554m^3-$
$55355m^4-m^5, 398581+55949982m+413000631m^2+421686548m^3+60344691m^4+$
$498222m^5+m^6\}$

vv:=

Transpose[

Table $\left[\text{SeriesCoefficient}\left[\{1, -7-m, 61+74m+m^2, -547-2739m-681m^2-m^3, 4921+80788m+85038m^2+6148m^3+m^4, -44287-2169797m-6590134m^2-2324554m^3-55355m^4-m^5, 398581+55949982m+413000631m^2+421686548m^3+60344691m^4+498222m^5+m^6\}, \{m,0,n\}\right], \{n,0,6\}\right]]$

vv//MatrixForm

vvv:=Table $\left[(-1)^{n+1}\text{vv}[[n,\text{All}]], \{n,1,7\}\right]$ **//MatrixForm**

vvv

Table $\left[(-1)^{n+1}\text{vv}[[n,\text{All}]], \{n,1,7\}\right]$ **.Table[1,** $\{n,0,6\}$**]**

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & -1 & 0 & 0 & 0 & 0 & 0 \\ 61 & 74 & 1 & 0 & 0 & 0 & 0 \\ -547 & -2739 & -681 & -1 & 0 & 0 & 0 \\ 4921 & 80788 & 85038 & 6148 & 1 & 0 & 0 \\ -44287 & -2169797 & -6590134 & -2324554 & -55355 & -1 & 0 \\ 398581 & 55949982 & 413000631 & 421686548 & 60344691 & 498222 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 61 & 74 & 1 & 0 & 0 & 0 & 0 \\ 547 & 2739 & 681 & 1 & 0 & 0 & 0 \\ 4921 & 80788 & 85038 & 6148 & 1 & 0 & 0 \\ 44287 & 2169797 & 6590134 & 2324554 & 55355 & 1 & 0 \\ 398581 & 55949982 & 413000631 & 421686548 & 60344691 & 498222 & 1 \end{pmatrix}$$

$\{1, 8, 136, 3968, 176896, 11184128, 951878656\}$

The code for section the inverse transformation of the embeddded submatrix to get the row sums for $m=1$ multiplied by $(-1)^{n+1}$ resulting to the sequence corresponding to **A062197.**

A**:=Table** $\left[\frac{n!}{k!}\text{SeriesCoefficient}\left[\left(\frac{1}{(1+z)^3}\right)\left(\frac{z}{1+z}\right)^k, \{z,0,n\}\right], \{n,0,6\}, \{k,0,6\}\right]$

AA:=Table $\left[(-1)^{n+1}A[[n,\text{All}]], \{n,1,7\}\right]$ **//MatrixForm**

AA

Table $\left[(-1)^{n+1}A[[n,\text{All}]],\{n,1,7\}\right]$ **.Table[1, {n, 0, 6}]**

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 12 & -8 & 1 & 0 & 0 & 0 & 0 \\ 60 & -60 & 15 & -1 & 0 & 0 & 0 \\ 360 & -480 & 180 & -24 & 1 & 0 & 0 \\ 2520 & -4200 & 2100 & -420 & 35 & -1 & 0 \\ 20160 & -40320 & 25200 & -6720 & 840 & -48 & 1 \end{pmatrix}$$

$\{1, 2, 5, 14, 37, 34, -887\}$

The transfer function and plot for the Bessel filter using $\left[\frac{1}{\sqrt[4]{1-4t}}, 1-\sqrt[4]{1-4t}\right]$

BodePlot[TransferFunctionModel[{{{1}}, 1s + 1}, s]]

BodePlot[TransferFunctionModel[{{{5}}, $s^2 + 5s + 5$}, s]]

BodePlot[TransferFunctionModel[{{{45}}, $s^3 + 12s^2 + 45s + 45$}, s]]

BodePlot[TransferFunctionModel[{{{585}}, $s^4 + 22s^3 + 177s^2 + 585s + 585$}, s]]

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